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CHARACTERIZATIONS OF SOME CLASSES OF Γ–SEMIGROUPS

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ABSTRACT. The author obtains ideal-theoretical characterizations of the following two classes of Γ–semigroups; (1) regular Γ–semigroups; (2) Γ-semigroups that are both regular and intra-regular.

In 1981, M. K. Sen([7]) introduced the concept of Γ-semigroup and M. K. Sen and N. K Saha ([8,9]) obtained some interesting results S. Lajos ([5,6]) gave some characterizations of regular and/or intra-regular semigroups. In this paper we proved some characterizations of Γ-semigroup by similar methods.

Let M and Γ be non-empty sets. Then M is called a Γ*–semigroup* if the following conditions hold :

(1) $a\alpha b \in M$, and $\alpha a\beta \in \Gamma$ for all $\alpha, \beta \in \Gamma$ and $a, b \in M$;

(2) $(a\alpha b)\beta c = a(\alpha b\beta)c = a\alpha(b\beta c)$ for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$.

For $A, B \subseteq M$, let $A\Gamma B = \{a\gamma b | a \in A, b \in B, \gamma \in \Gamma\}$.

EXAMPLE. Let M be the set of all integers of the form $4n+1$ where n is an integer and let Γ be the set of all integers of the form $4n+3$. If $a\alpha b$ is $a+\alpha+b$ and $\alpha a\beta$ is $\alpha+a+\beta$(usual sum of the integers) for all $a, b \in M$ and $\alpha, \beta \in \Gamma$, then M is a Γ-semigroup.

DEFINITION 1 [8,9]. An element a of a Γ-semigroup M is called *regular* if $a \in a\Gamma M\Gamma a$. A Γ-semigroup M is called *regular* if every element of M is regular .

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DEFINITION 2. A Γ-subsemigroup T is called *intra-regular* if, for all $a \in T$, there exist $x, y \in T$ such that $a \in x\Gamma a\Gamma a\Gamma y$.

DEFINITION 3 [8,9]. Let M be a Γ-semigroup . A non-empty subset B of M is said to be a *Γ-subsemigroup* of M if $B\Gamma B \subseteq B$.

DEFINITION 4. Let M be a Γ-semigroup. A nonempty subset I of M is said to be *right(left) ideal* of M if $I\Gamma M \subseteq I(M\Gamma I \subseteq I)$.

If I is both a right ideal and a left ideal then we say that I is an ideal of M.

DEFINITION 5. A non-empty subset Q of the Γ-semigroup M is called a *quasi-ideal* of M if $Q\Gamma M \cap M\Gamma Q \subseteq Q$.

Every left (resp. right) ideal is a quasi-ideal Also every ideal is a quasi-ideal.

DEFINITION 6. Let B be a non-empty subset of a $\Gamma-$semigroup M. The set B is called a *bi-ideal* of M if $B\Gamma M\Gamma B \subseteq B$

Every quasi-ideal is a bi-ideal.

First we give a new characterization of regular Γ-semigroups.

THEOREM 7. *A Γ-semigroup M is regular if and only if the inclusion*

$$(1) \qquad B \cap I \cap L \subseteq B\Gamma I\Gamma L$$

holds for every bi-ideal B, every left ideal L, and every two-sided ideal I of M, provided that the intersection $B \cap I \cap L$ is non-empty.

Proof. Let M be a Γ-semigroup and let a be an element of $B \cap I \cap L$, where B is a bi-ideal, L is a left ideal, and I is a two-sided ideal of M. Then there exists an element x in M such that

$$\begin{aligned} a &= a\gamma x\mu a \\ &= a\gamma x\mu(a\gamma x\mu a) \\ &= a\gamma x\mu a\gamma x\mu a\gamma x\mu a \\ &= (a\gamma x\mu a)\gamma(x\mu a)\gamma(x\mu a) \\ &\in B\Gamma I\Gamma L \end{aligned}$$

for some γ, μ in Γ. Hence the condition (1) holds.

Conversely, if M is a Γ-semigroup with property (1), then we get

$$(2) \qquad R \cap M \cap L \subseteq R\Gamma M\Gamma L \subseteq R\Gamma L$$

for every left ideal L and every right ideal R of M. Therefore M is regular, indeed.

THEOREM 8. *A Γ-semigroup M is both regular and intra-regular if and only if the inclusion*

$$(3) \qquad B \cap L \subseteq B\Gamma L\Gamma B$$

holds for every bi-ideal B and every left ideal L of M with $B \cap L \neq \emptyset$

Proof. Let M be a regular and intra-regular Γ-semigroup. Then for every element a of M there exist elements $x, y, z \in M$ such that

$$(4) \qquad a = a\gamma x\mu a = y\delta a\beta a\nu z$$

for some $\gamma, \mu, \delta, \beta$ and $\nu \in \Gamma$.

If $a \in B \cap L$, where B is a bi-ideal, L is a left ideal of M, we have elements x, y, z in M such that

$$\begin{aligned}(5) \qquad a &= (a\gamma x)\mu a = a\gamma x\mu(a\gamma x\mu a)\\ &= a\gamma x\mu(y\delta a\beta a\nu z\gamma x\mu a)\\ &= (a\gamma x\mu a)\gamma(x\mu y\delta a)\beta(a\nu z\gamma x\mu a)\\ &\in B\Gamma L\Gamma B.\end{aligned}$$

Conversely, if M is a Γ-semigroup with property (3), then (3) implies

$$(6) \qquad L \cap R \subseteq R\Gamma L\Gamma R \subseteq L\Gamma R$$

for every left ideal L and every right ideal R of M. By Theorem 2 in [5], M is intra-regular. In this case of $L = M$, the inclusion (3) implies

$$(7) \qquad B \subseteq B\Gamma M\Gamma B$$

for every bi-ideal B of M. Hence we get $B = B\Gamma M\Gamma B$, and thus M is a regular Γ-semigroup (cf. [5], Theorem 1).

THEOREM 9. *For a Γ-semigroup M the following conditions are pairwise equivalent:*

(1) *M is regular and intra-regular.*

(2) *For every bi-ideal B and every left ideal L of M,*

$$B \cap L \subseteq B\Gamma L\Gamma B.$$

(3) *For every bi-ideal B and every right ideal R of M,*

$$B \cap R \subseteq B\Gamma R\Gamma B.$$

(4) *For every left ideal L and every quasi-ideal Q of M,*

$$L \cap Q \subseteq Q\Gamma L\Gamma Q.$$

(5) *For every right ideal R and every quasi-ideal Q of M,*

$$Q \cap R \subseteq Q\Gamma R\Gamma Q.$$

(6) *For every bi-ideal B and every quasi-ideal Q of M,*

$$B \cap Q \subseteq B\Gamma Q\Gamma B.$$

(7) *For every bi-ideal B and every quasi-ideal Q of M,*

$$B \cap Q \subseteq Q\Gamma B\Gamma Q.$$

The Proof of this result is similar to that of Theorem 8, and we omit it.

REMARK. It is easy to see Theorem 7 remain true with generalized bi-ideal or quasi-ideal instead of bi-ideal.

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