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HOLOMORPHIC LINE BUNDLES OF COHOMOLOGY GROUPS FOR A COMPLEX TORUS

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1. Introduction

Let Γ be a discrete subgroup of $\mathbf{C}^n$. Then we can construct a complex Lie group $\mathbf{T}^n$ from the subgroup Γ. Hence we construct a cohomology group for the structure sheaf O of $\mathbf{T}^n$. In the case of weakly pseudoconvex manifolds the $\bar{\partial}$-problem depends not only on boundary conditions, but also on complex structures (see [2,3,8]). H. Grauert [1] showed that there exists a C^∞ weakly pluriharmonic exhaustion function on a Picard set. H. Kazama and K. H. Shon [4,5] obtained a criterion for the $\bar{\partial}$-cohomology in the Picard group, using the theory of Diophantine approximation. In this paper we investigate the properties of weakly pseudoconvex manifolds not containing the strictly pseudoconvex manifold and holomorphic line bundles of cohomology groups for a complex torus

2. Preliminaries

Let $\mathbf{T}^q$ be a complex torus of complex dimension q and Γ be a

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discrete subgroup of $\mathbf{C}^q$, that is,

$$\begin{aligned}&\Gamma(u_1, u_2, \cdots, u_q, u_{q+1}, \cdots, u_{2q})\\ &= \{m_1u_1 + m_2u_2 + \cdots + m_qu_q + m_{q+1}u_{q+1} + \cdots + m_{2q}u_{2q}\\ &\qquad : \forall m_i \in \mathbf{Z}, 1 \le i \le 2q\}.\end{aligned}$$

Then $\mathbf{T}^q = \mathbf{C}^q/\Gamma(u_1, u_2, \cdots, u_{2q})$ is a compact complex Lie group.

DEFINITION 2.1. *A manifold X of complex dimension n is said to be a strictly pseudoconvex manifold if there exists a C^∞ function $\varphi : X \to \mathbf{R}$ such that*

(1) *The Levi form $[\frac{\partial^2\varphi}{\partial z_i \partial \bar{z}_j}]$ is everywhere positive.*

(2) $X_c := \{x \in X : \varphi(x) < c\} \subset\subset X, \forall c \in \mathbf{R}$.

DEFINITION 2.2. *A manifold X of dimension n is said to be a weakly pseudoconvex manifold (or weakly 1-complete manifold) if there exists a C^∞ function $\varphi : X \to \mathbf{R}$ such that*

(1) *The Levi form is everywhere positive semi-definite.*

(2) $X_c \subset\subset X, \forall c \in \mathbf{R}$.

Consider an exact sequence of sheaves

$$\begin{aligned}&0 \to \mathbf{Z} \xrightarrow{i} O \xrightarrow{\Phi} O^* \to 0,\\ &0 \to \mathbf{Z} \to \mathcal{C} \to \mathcal{C}^* \to 0,\end{aligned}$$

where $\Phi(f_x) = e^{2\pi\sqrt{-1}f_x}$, $f_x \in O$, O is the sheaf of germs of complex-valued C^∞ functions, O^* is the nonzero sheaf, C is the sheaf of germ of continuous functions and C^* is the nonzero sheaf. Let $H^1(\mathbf{T}^q, O^*)$ be the cohomology group of all holomorphic line bundles on $\mathbf{T}^q$. Then we have the following exact sequences

$$\begin{aligned}&0 \to H^0(\mathbf{T}^q, \mathbf{Z}) \to H^0(\mathbf{T}^q, O) \to H^0(\mathbf{T}^q, O^*)\\ &\quad \to H^1(\mathbf{T}^q, \mathbf{Z}) \to H^1(\mathbf{T}^q, O) \xrightarrow{\Phi} H^1(\mathbf{T}^q, O^*) \to H^2(\mathbf{T}^q, \mathbf{Z}) \to \cdots,\end{aligned}$$

and

$$\cdots \to H^1(\mathbf{T}^q, \mathbf{Z}) \to H^1(\mathbf{T}^q, C) \to H^1(\mathbf{T}^q, C^*) \to H^2(\mathbf{T}^q, \mathbf{Z}) \to \cdots .$$

Since the holomorphic line bundle $L \in H^1(\mathbf{T}^q, O^*)$ is topological trivial if and only if the first Chern class $c_1(L) = 0$, we have the group of all topological holomorphic line bundles $\mathfrak{F}$,

$$\begin{aligned}\mathfrak{F} &= \{L \in H^1(\mathbf{T}^q, O^*) : c_1(L) = 0\} \\ &= Im\ \Phi \\ &\cong H^1(\mathbf{T}^q, O)/Ker\ \Phi \\ &= H^1(\mathbf{T}^q, O)/H^1(\mathbf{T}^q, \mathbf{Z}) \\ &=: P^0_{ic}(\mathbf{T}^q).\end{aligned}$$

On the complex torus, we have

$$dim_{\mathbf{C}}\ H^1(\mathbf{T}^q, O) = q$$

Hence

$$H^1(\mathbf{T}^q, O) = H^1(\mathbf{T}^o, \{m_1 u_1 + \cdots + m_{2q} u_{2q} : m_i \in \mathbf{Z}\}) \cong \mathbf{C}^q$$

and

$$\begin{aligned}P^0_{ic}(\mathbf{T}^q) &= H^1(\mathbf{T}^q, O)/H^1(\mathbf{T}^q, \mathbf{Z}) \\ &\cong \mathbf{C}^q/\Gamma(u_1, u_2, \cdots, u_{2q}) \\ &= \text{a complex q - dimensional torus,}\end{aligned}$$

in the sense of complex space. Thus we have the following lemma.

LEMMA 2.3. *Let $\mathfrak{F}$ be the group of holomorphic line bundles on $\mathbf{T}^q$ with the first Chern class zero. Then $\mathfrak{F}$ is a family of weakly pseudo-convex manifold.*

Let $\{U_i\}_{i \in I}$ be an open covering of $\mathbf{T}^q$ and $L = \cup_i \{U_i \times \mathbf{C}\}/\backsim$, where $\cup$ is a disjoint union and $\backsim$ is an equivalence relation. Then for a projection $\pi : L \to \mathbf{T}^q$, we have an isomorphism $\varphi_i : \pi^{-1}(U_i) \cong U_i \times \mathbf{C}$ and there exists a family $\{f_{ij}\} \in H^1(\{U_i\}, O^*)$.

PROPOSITION 2.4. *If* $\{f_{ij}\} \in H^1(\{U_i\}, O^*)$, *then* $|f_{ij}(x)| \equiv 1$.

Proof. From the properties of a compact Käller manifold X of A. Morrow - K. Kodaira [7], we have an (1, 1)-form φ on X such that for a C^∞, 1-form ψ, satisfying $\varphi = d\psi$. Hence there is a C^∞-form f on X with $\varphi = \partial\bar{\partial} f$. By Lemma 2.3, the topological trivial holomorphic line bundle L on $\mathbf{T}^q$ is weakly pseudoconvex. By H. Kazama and T. Umeno [6], we have $|f_{ij}| \equiv 1$.

THEOREM 2.5. *If the holomorphic line bundle* $L \in H^1(\mathbf{T}^q, O^*)$ *is topological trivial, then the bundle* L *is weakly 1-complete.*

Proof. By Proposition 2.4, we have $\{f_{ij}\} \in H^1(\mathbf{T}^q, O^*)$ satisfying $|f_{ij}| \equiv 1$. We define a mapping $\varphi : L \to \mathbf{R}$ as follows. For $p \in L$ and an open covering $\{U_i\}$ of $\mathbf{T}^q$, there exists $i \in \mathbf{I}$ such that $p \in \pi^{-1}(U_i)$. Hence there exists a biholomorphic mapping

$$\varphi : \pi^{-1}(U_i) \to U_i \times \mathbf{C}$$

satisfying $\varphi(p) = (\pi(p), z_i(p)) \in U_i \times \mathbf{C}$. We define $\varphi(p) := |z_i(p)|^2 \geq 0$. Then $L_c := \{p \in L \mid \varphi(p) < c\} \subset\subset L$. In fact, since the torus $\mathbf{T}^q$ is compact, we have a finite open covering $\{U_i\}_{i=1}^k$ of $\mathbf{T}^q$. Therefore

$$\begin{aligned} L_c \cap \pi^{-1}(U_i) &= \{p \in \pi^{-1}(U_i) : |z_i(p)|^2 < c, \varphi_i(p) = (\pi(p), z_i(p)\} \\ &= \varphi_i^{-1}(U_i \times \{z : |z|^2 < c\}) \\ &\subset\subset L. \end{aligned}$$

Thus, we have

$$L_c = \cup_{k=1}^n (L_c \cap \pi^{-1}(U_i)) \subset\subset L.$$

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