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A NOTE ON THE TWO DIMENSIONAL SURFACE IN FOUR DIMENSIONAL EQUIAFFINE SPACE

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Abstract. In this paper, we investigate the existence of the two dimensional surface in four dimensional equiaffine space and characterize that surface

0. Introduction

In [7-9] an invariant clothing of the families of two dimensional and m dimensional planes in four dimensional and $2m$ dimensional equiaffine space ($m > 2$) respectively, has been carried out. By this construction the case when the family of two dimensional planes envelops some two dimensional surface in the four dimensional space A_4 and the family of m dimensional planes envelops some m dimensional surfaces in A_{2m}, is taken out of consideration

This article is devoted to an invariant construction of clothings of the families of two dimensional planes in A_4 enveloping some two dimensional general surface S_2 that is, in fact, reduced to the study of the two dimensional surface in four dimensional equiaffine space. Therefore, this article is referred to the General Theory of surfaces(see references [3,4,6]).

In this article §1 is devoted to analytical apparatus, in which, in particular, an analytical construction of the canonical frame of the surface S_2 in A_4 is brought, in §2 a Basic Theory of the Affine Theory of surfaces in A_4 is proved. In §3 the clothing of a surface S_2 is built

Notation and terminology correspond to the adopted in [1-9].

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1. An analytical fixation of the frame

The surface S_2 in A_4 is a hodograph of the vector function of two arguments: $\vec{r} = \vec{r}(u, v)$.

If ω^i and ω_i^k are the Pfaff's forms from these arguments, the derivative formulas of a mobile reper are to be expressed as

$$d\vec{r} = \omega^i \vec{e}_i \ , \ d\vec{e}_i = \omega_i^k \vec{e}_k, \tag{1}$$

where the forms ω^i , ω_i^k satisfy the structural equations

$$D\omega^i = \omega^j \wedge \omega_j^i, \ D\omega_i^k = \omega_i^j \wedge \omega_j^k. \tag{2}$$

Here the formulas $\omega_1^1, \omega_2^2, \omega_3^3, \omega_4^4$ by virtue of the condition of equiaffinity

$$(\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4) = 1,$$

satisfy the differential correlation $\omega_1^1 + \omega_2^2 + \omega_3^3 + \omega_4^4 = 0$.

When the frame changes, the main parameters u, v remain constant but the secondary parameters, according to which the Pfaff's forms are denoted usually as π^i, π_i^k and differentiation as δ, change. We have

$$\delta\vec{r} = \pi^i \vec{e}_i, \ \delta\vec{e}_i = \pi_i^k \vec{e}_k.$$

On placing the beginning of frame in a point of the surface, we obtain

$$\delta\vec{r} = 0, \ \pi^1 = \pi^2 = \pi^3 = \pi^4 = 0.$$

The forms $\omega^1, \omega^2, \omega^3, \omega^4$ have become the main ones and they depend only on two arguments u and v. Consequently, there are two linear dependences between them. Choosing them in the form

$$\omega^3 = 0, \ \omega^4 = 0, \tag{3}$$

we have $(d\vec{r}, \vec{e}_1, \vec{e}_2) = 0$, that is, the vectors $\vec{e}_1$ and $\vec{e}_2$ will lie inside the tangent plane.

To continue the construction of the reper we differentiate (3) externally,using (2),we obtain $\omega^1 \wedge \omega_1^{\hat{\alpha}} + \omega^2 \wedge \omega_2^{\hat{\alpha}} = 0$, $(\hat{\alpha} = 3, 4)$.

Hence, according to the Cartan's lemma,

$$\omega_{\alpha}^{\hat{\alpha}} = A_{\alpha\beta}^{\hat{\alpha}}\omega^{\beta},\ A_{[\alpha\beta]}^{\hat{\alpha}} = 0,\ (\alpha,\beta,\gamma = 1,2,\ \hat{\alpha},\hat{\beta},\hat{\gamma} = 3,4). \tag{4}$$

Differentiating these differential equations externally, we obtain

$$(dA_{\alpha\beta}^{\hat{\alpha}} - A_{\gamma\beta}^{\hat{\alpha}}\omega_{\alpha}^{\gamma} - A_{\alpha\gamma}^{\hat{\alpha}}\omega_{\beta}^{\gamma} + A_{\alpha\beta}^{\hat{\beta}}\omega_{\hat{\beta}}^{\hat{\alpha}}) \wedge \omega^{\beta} = 0. \tag{5}$$

Let us consider two linear elements
1. $\omega^{\alpha}(d)$, $\omega_i^k(d)$ and $\omega^{\hat{\alpha}}(d) = 0$,
2. $\omega^{\alpha}(\delta)$ and $\omega_i^k(\delta) = \pi_i^k$.

Substituting them in (5), we obtain

$$(\delta A_{\alpha\beta}^{\hat{\alpha}} - A_{\gamma\beta}^{\hat{\alpha}}\pi_{\alpha}^{\gamma} - A_{\alpha\gamma}^{\hat{\alpha}}\pi_{\beta}^{\gamma} + A_{\alpha\beta}^{\hat{\beta}}\pi_{\hat{\beta}}^{\hat{\alpha}}) = 0 \tag{6}$$

The following fixation is possible with the use of (6).

$$\begin{gathered} A_{12}^3 = 0, A_{12}^4 = 0, A_{11}^4 = 0, \qquad A_{22}^3 = 0, A_{11}^3 = 1, A_{22}^4 = 1, \\ 2\pi_1^1 - \pi_3^3 = 0,\ \pi_2^1 = 0,\ \pi_4^3 = 0, 2\pi_2^2 - \pi_4^4 = 0,\ \pi_1^2 = 0,\ \pi_3^4 = 0. \end{gathered} \tag{9}$$

From (4)

$$\omega_1^3 = \omega^1,\ \omega_2^3 = 0,\ \omega_1^4 = 0,\ \omega_2^4 = \omega^2, \tag{8}$$

after applying the Cartan's lemma, formulas (5) brought to the correlations.

$$\begin{aligned} 2\omega_1^1 - \omega_3^3 &= A\omega^1 + B\omega^2, \\ 2\omega_2^2 - \omega_4^4 &= A^*\omega^2 + B^*\omega^1, \\ \omega_2^1 &= B\omega^1 + C\omega^2, \\ \omega_1^2 &= B^*\omega^2 + C^*\omega^1, \\ \omega_4^3 &= -C\omega^1 + E\omega^2, \\ \omega_3^4 &= -C^*\omega^2 + E^*\omega^1. \end{aligned} \tag{9}$$

We differentiate (9) externally, using (2) and (8). Then, we have

(10)
$$(dA - A\omega_1^1 - B\omega_1^2 + 3\omega_3^1 + (EE^* + CC^* - 2BB^*)\omega^2) \wedge \omega^1$$
$$+(dB - B\omega_2^2 - A\omega_2^1) \wedge \omega^2 = 0,$$
$$(dB^* - B^*\omega_1^1 - A^*\omega_1^2)\wedge\omega^1 +$$
$$(dA^* - A^*\omega_2^2 - B^*\omega_2^1 + 3\omega_4^2 + (2BB^* - EE^* - CC^*)\omega^1) \wedge \omega^2 = 0,$$
$$(dB - B\omega_2^2 - C\omega_1^2) \wedge \omega^1 + (dC + C\omega_1^1 - B\omega_2^1 - 2C\omega_2^2 + \omega_4^1) \wedge \omega^2 = 0,$$
$$(dC^* + C^*\omega_2^2 - B^*\omega_1^2 - 2C^*\omega_1^1 + \omega_3^2) \wedge \omega^1$$
$$+(dB^* - B^*\omega_1^1 - C^*\omega_2^1) \wedge \omega^2 = 0,$$
$$(-dC + C\omega_1^1 - E\omega_1^2 - \omega_4^1 - C\omega_3^3 + C\omega_4^4) \wedge \omega^1$$
$$+(dE - E\omega_2^2 + C\omega_2^1 + E\omega_3^3 - E\omega_4^4) \wedge \omega^2 = 0,$$
$$(dE^* - E^*\omega_1^1 + C^*\omega_1^2 - E^*\omega_3^3 + E^*\omega_4^4) \wedge \omega^1$$
$$+(-dC^* + C^*\omega_2^2 - E^*\omega_2^1 - \omega_3^2 + C^*\omega_3^3 - C^*\omega_4^4) \wedge \omega^2 = 0.$$

Hence, in a similar manner as above, we shall arrive at the following correlations for fixation of the rest of secondary forms.

$$\delta A - A\pi_1^1 + 3\pi_3^1 = 0, \qquad \delta B - B\pi_2^2 = 0,$$
$$\delta B^* - B^*\pi_1^1 = 0, \quad \delta A^* - A^*\pi_2^2 + 3\pi_4^2 = 0,$$
$$\delta C + 3C\pi_1^1 + \pi_4^1 = 0, \quad \delta C^* - 3C^*\pi_2^2 + \pi_3^2 = 0,$$
$$\delta E - 5E\pi_2^2 = 0, \qquad \delta E^* - 5E^*\pi_1^1 = 0. \tag{11}$$

Using correlations (11), we carry out the following final fixation of the affine frame on the surface S_2.

$$A = 0, B = 1, A^* = 0, B^* = 1, C = 0, C^* = 0,$$
$$\pi_1^1 = 0, \pi_2^2 = 0, \pi_3^1 = 0, \pi_3^2 = 0, \quad \pi_4^1 = 0, \ \pi_4^2 = 0. \tag{12}$$

From (9) we obtain

$$\omega_2^1 = \omega^1, \qquad \omega_1^2 = \omega^2, \qquad \omega_4^3 = E\omega^2, \quad \omega_3^4 = E^*\omega^1,$$
$$2\omega_1^1 - \omega_3^3 = \omega^2, \quad \omega_2^2 - \omega_4^4 = \omega^1. \tag{13}$$

Formulas (10), by virtue of (13) and (12), after applying the Cartan's lemma, brought to the differential equations.

$$(14) \qquad \omega^{\alpha}_{\hat{\alpha}} = A^{\alpha}_{\hat{\alpha}\beta}\omega^{\beta}, \ \omega^{i}_{\alpha} = A^{\alpha}_{i\beta}\omega^{\beta}$$

(do not sum in α), where

(15)

$$\begin{aligned}
(dE^* + E^*\omega^2 + A^2_{31}\omega^2 - E^* A^3_{32}\omega^2 + 2E^* A^2_{22}\omega^2 - E^* A^1_{12}\omega^2) \wedge \omega^1 = 0,\\
(dE + E\omega^1 - A^1_{42}\omega^1 - EA^4_{41}\omega^1 + 2EA^1_{11}\omega^1 - EA^2_{21}\omega^1) \wedge \omega^2 = 0,\\
A^2_{22} + A^1_{41} = 1, \ A^1_{11} + A^2_{32} = 1, \ A^3_{32} = 2A^1_{12} - 1, A^4_{41} = 2A^2_{21} - 1,\\
-3A^1_{32} + EE^* - A^2_{21} + 3 = 0, \ -3A^2_{41} + EE^* + 3 - A^1_{12} = 0.
\end{aligned}$$

2. The basic theorem of the affine theory of surfaces in A_4

As all the secondary forms have been reduced to zero, the forms ω^k_i are the linear combinations of the forms ω^1 and ω^2, which are defined due to formulas (3),(8), (13) and (14). In these combinations the correspoding coefficients satisfy not only correlations (15), but also the structural equations, which follow from (2). Let us notice that equations (13) are derived from (8), and (14) are derived from (13) by means of external differentiation.

THEOREM 1. *The surface S_2 in A_4 exists and is determined with arbitrariness of two functions of two arguments.*

Proof. Differentiating (14) externally, we obtain

(16)

$$\begin{aligned}
dA^1_{31} \wedge \omega^1 + dA^1_{32} \wedge \omega^2 =\\
(2A^1_{31} - A^2_{32} + E^* A^1_{42} - 2A^2_{21}A^1_{32} - 2A^1_{11}A^1_{32} - 2A^1_{12}A^1_{31})\omega^1 \wedge \omega^2,\\
dA^2_{41} \wedge \omega^1 + dA^2_{42} \wedge \omega^2 =
\end{aligned}$$

$$(2A_{42}^2 - A_{41}^1 + EA_{31}^2 - 2A_{12}^1 A_{41}^2 - 2A_{22}^2 A_{41}^2 - 2A_{21}^2 A_{42}^2)\omega^2 \wedge \omega^1,$$

$$dA_{31}^2 \wedge \omega^1 + dA_{32}^2 \wedge \omega^2 =$$

$$(1 + 2(A_{31}^2 - A_{12}^1) + A_{12}^1(2A_{21}^2 - 3A_{31}^2) + A_{32}^2(1 - A_{11}^1 - 4A_{21}^2)$$

$$-A_{21}^2 + A_{31}^1 + A_{22}^2 A_{31}^2 + E^* A_{42}^2)\omega^1 \wedge \omega^2,$$

$$dA_{41}^1 \wedge \omega^1 + A_{42}^1 \wedge \omega^2 = (2A_{42}^1 + A_{42}^1(A_{11}^1 - 3A_{21}^2)$$

$$+A_{42}^2 + EA_{31}^1 - A_{41}^1(3A_{12}^1 + A_{22}^2))\omega^2 \wedge \omega^1,$$

$$dA_{11}^1 \wedge \omega^1 + dA_{12}^1 \wedge \omega^2 =$$

$$(-1 + A_{32}^1 + A_{11}^1 - A_{12}^1 - A_{11}^1 A_{12}^1 + A_{21}^2 A_{12}^1),$$

$$dA_{21}^2 \wedge \omega^1 + dA_{22}^2 \wedge \omega^2 =$$

$$(-1 + A_{22}^2 - A_{21}^2 + A_{41}^2 - A_{22}^2 A_{21}^2 + A_{21}^2 A_{12}^1)\omega^2 \wedge \omega^1.$$

Thus, if the given functions

$$A_{31}^1,\ A_{32}^1,\ A_{31}^2,\ A_{32}^2,\ A_{41}^1,\ A_{42}^1,\ A_{41}^2,\ A_{42}^2,\ A_{11}^1,\ A_{12}^1,$$
$$A_{21}^2,\ A_{22}^2,\ E,\ E^*,$$

satisfy (15) and (16), then affinity of the surface S_2 in A_4 is given.

The definition of this surface is reduced to integration of the following quite integrable system of derivative equations

(17)

$$d\vec{r} = \omega^1\vec{e}_1 + \omega^2\vec{e}_2,$$
$$d\vec{e}_1 = (A_{11}^1\omega^1 + A_{12}^1\omega^2)\vec{e}_1 + \omega^2\vec{e}_2 + \omega^1\vec{e}_3,$$
$$d\vec{e}_2 = \omega^1\vec{e}_1 + (A_{21}^2\omega^1 + A_{22}^2\omega^2)\vec{e}_2 + \omega^2\vec{e}_4,$$
$$d\vec{e}_3 = (A_{31}^1\omega^1 + A_{32}^1\omega^2)\vec{e}_1 + (A_{31}^2\omega^1 + A_{32}^2\omega^2)\vec{e}_2 + \omega_3^3\vec{e}_3 + E^*\omega^1\vec{e}_4,$$
$$d\vec{e}_4 = (A_{41}^1\omega^1 + A_{42}^1\omega^2)\vec{e}_1 + (A_{41}^2\omega^1 + A_{42}^2\omega^2)\vec{e}_2 + E\omega^2\vec{e}_3 + \omega_4^4\vec{e}_4,$$
$$\omega_3^3 = 2A_{11}^1\omega^1 + (2A_{12}^1 - 1)\omega^2,$$
$$\omega_4^4 = 2A_{22}^2\omega^2 + (2A_{21}^2 - 1)\omega^1.$$

For the system (17) according to Bachvalov's Theorem, we obtain that the arbitrariness of the existence of surface S_2 in A_4 is equal to two functions of two arguments.

REMARK. *The same arbitrariness is to be received on considering (15)–(16).*

3. The focal(conjugate) lines on the surface S_2

The focal hyperplanes. Rationing of the vectors $\vec{e}_3$ and $\vec{e}_4$.

3.1. The focal straight lines l_1 and l_2

We shall put

$$L_2 = (\vec{A}, \vec{e}_1, \vec{e}_2) = (\vec{r}, \vec{e}_1, \vec{e}_2) \tag{18}$$

the tangent plane to S_2 at the point $\vec{A}$.

Let the point in L_2 with the radius vector

$$\vec{X} = \vec{r} + x^\alpha \vec{e}_\alpha \in L_2 \tag{19}$$

be a focus, that is, describes a line with a tangent belonged to L_2 along a (focal) line on S_2 [8].Then $(d\vec{X}, \vec{e}_1, \vec{e}_2) = 0$, which, by virtue of (1), (3) and (4),leads to the correlations.

$$x^\alpha A^{\hat{\alpha}}_{\alpha\beta} \omega^\beta = 0, \ (\alpha, \beta = 1, 2; \ \hat{\alpha}, \hat{\beta} = 3, 4) \tag{20}$$

This system has the non-trivial solutions in ω^α if and only if

$$\begin{aligned} &det[x^\alpha A^{\hat{\alpha}}_{\alpha\beta}] \\ &= \begin{vmatrix} x^\alpha A^3_{\alpha 1} & x^\alpha A^3_{\alpha 2} \\ x^\beta A^4_{\beta 1} & x^\beta A^4_{\beta 2} \end{vmatrix} \\ &= (A^3_{\alpha 1} A^4_{\beta 2} - A^3_{\alpha 2} A^4_{\beta 1}) x^\alpha x^\beta \\ &= (A^3_{11} A^4_{12} - A^3_{12} A^4_{11})(x^1)^2 + (A^3_{11} A^4_{22} - A^3_{22} A^4_{11}) x^1 x^2 \\ &\quad + (A^3_{21} A^4_{22} - A^4_{21} A^3_{22})(x^2)^2 = 0. \end{aligned} \tag{21}$$

Thus, to each point $\vec{A} \in S_2$ in A_4 the plane L_2 correspond two focal lines l_1 and l_2 The tangent line (focal or conjugate) at the point $\vec{A}$ on S_2 corresponds to each of such straight lines by virtue of (20)

3.2. The focal hyperplanes Γ_3^1 and Γ_3^2

Let us take up the hyperplane at each point $\vec{A} \in S_2$

$$x: \ x_{\hat{\alpha}} x^{\hat{\alpha}} = 0, \tag{22}$$

passing through L_2. Let this hyperplane be a focal hyperplane, that is, the plane containing L_2 which is near it along some (focal) line on S_2 [8]. From $d(\vec{A}, \vec{e}_1, \vec{e}_2) = (...)^{\alpha} \vec{e}_{\alpha} + \omega_1^{\hat{\alpha}} (\vec{A}, \vec{e}_{\hat{\alpha}}, \vec{e}_2) + \omega_2^{\hat{\alpha}} (\vec{A}, \vec{e}_1, \vec{e}_{\hat{\alpha}})$ by virtue of (22), we obtain

$$x_{\hat{\alpha}} A^{\hat{\alpha}}_{\alpha\beta} \omega^{\beta} = 0. \tag{23}$$

This system has the non–trivial solutions in ω^{α} if and only if
(24)

$$\begin{aligned}
&det[x_{\hat{\alpha}} A^{\hat{\alpha}}_{\alpha\beta}] \\
&= \begin{vmatrix} x_{\hat{\alpha}} A^{\hat{\alpha}}_{11} & x_{\hat{\alpha}} A^{\hat{\alpha}}_{12} x_{\hat{\beta}} A^{\hat{\beta}}_{12} & x_{\hat{\beta}} A^{\hat{\beta}}_{22} \end{vmatrix} \\
&= (A^{\hat{\alpha}}_{11} A^{\hat{\beta}}_{22} - A^{\hat{\alpha}}_{12} A^{\hat{\beta}}_{12}) x_{\hat{\alpha}} x_{\hat{\beta}} \\
&= (A^3_{11} A^3_{22} - A^3_{12} A^3_{12})(x_3)^2 + (A^3_{11} A^4_{22} + A^3_{22} A^4_{11} - 2A^3_{12} A^4_{12}) x_3 x_4 \\
&\qquad + (A^4_{11} A^4_{22} - A^4_{12} A^4_{12})(x_4)^2 \\
&= 0.
\end{aligned}$$

Thus, to each point $\vec{A} \in S_2$ in A_4 correspond two focal (tangent) [9] hyperplanes Γ_3^1 and Γ_3^2, which, by virtue of (20) and (23) contain L_2 and L_2' along the corresponding focal (conjugate) lines on S_2. In this case the plane L_2 intersects with its contiguous L_2' along the corresponding focal line on S_2 with the tangent l_α in the straight line l_α.

3.3. The characteristics of the fixation carried out

From (20) and (24) we notice that the following expression will be the discriminant of these quadratic equations:

$$\begin{aligned}
\Delta = &(A^3_{11} A^4_{22} - A^3_{22} A^4_{11})^2 \\
&- 4(A^3_{21} A^4_{22} - A^4_{21} A^3_{22})(A^3_{11} A^4_{12} - A^3_{12} A^4_{11}).
\end{aligned} \tag{25}$$

In this paragraph one considers the case, when

$$\Delta \neq 0 \tag{26}$$

on the surfase S_2 in A_4. In this case each of quadratic equations (21) and (24) will have two different solutions.

It follws from (21)–(26) that the fixation of the affine frame $\{\vec{r}, \vec{e}_i\}$ of the surface S_2 in A_4, carried out according to formulas (7), is characterized by

$$\begin{gathered} l_1 = (\vec{A}, \vec{e}_1), \qquad l_2 = (\vec{A}, \vec{e}_2), \\ \Gamma_3^1 = (\vec{A}, \vec{e}_1, \vec{e}_2, \vec{e}_3), \quad \Gamma_3^2 = (\vec{A}, \vec{e}_1, \vec{e}_2, \vec{e}_4) \end{gathered} \tag{27}$$

so that, the focal lines on S_2 are the coordinate lines and the corresponding tangents:

$$l_1 : \ \omega^2 = 0, \ l_2 . \ \omega^1 = 0. \tag{28}$$

3.4. Focuses on the straight lines l_1 and l_2

Let the points

$$\vec{t} = \vec{A} + t\vec{e}_1 \in l_1, \ \vec{\tau} = \vec{A} + \tau\vec{e}_2 \in l_2 \tag{29}$$

be focuses of the lines l_1 and l_2 respectively. From $(d\vec{t}, \vec{e}_1) = 0$, $(d\vec{\tau}, \vec{e}_2) = 0$ by virtue of (1),(3),(8) and (9), we obtain

$$\begin{aligned} l_1 &: \omega^2 + t(C^*\omega^1 + B^*\omega^2) = 0, \\ & t\omega^1 = 0, \\ l_2 &: \omega^1 + \tau(B\omega^1 + C\omega^2) = 0, \\ & \tau\omega^2 = 0. \end{aligned} \tag{30}$$

Hence, the following formulas are focuses and focal directions

$$\begin{aligned} l_1 &: 1) \ \vec{A} : \ \omega^2 = 0, \\ & \quad 2) \ \vec{t}_1 = \vec{A} - \frac{1}{B^*}\vec{e}_1 : \ \omega^1 = 0; \\ l_2 &: 1) \ \vec{A} : \ \omega^1 = 0, \\ & \quad 2) \ \vec{\tau}_1 = \vec{A} - \frac{1}{B}\vec{e}_2 : \ \omega^2 = 0. \end{aligned} \tag{31}$$

It follows from (31) that by fixation (12) ($B = 1$, $B^* = 1$) the vectors $\vec{e}_1$ and $\vec{e}_2$ are normalized, so that the points

$$\vec{t_1} = \vec{A} - \vec{e}_1 \in l_1, \quad \vec{\tau_1} = \vec{A} - \vec{e}_2 \in l_2 \tag{32}$$

are focuses of the rays l_1 and l_2 respectively. Under the circumstances, the case when $B = 0$ ($resp. B^* = 0$), the focus $\tau_1(resp. t_1)$ is not an eigen point, that is, a congruence of the straight lines l_2 ($resp. l_1$) is cylindrical, is taken out of consideration.

3.5. The characteristics of the hyperplanes Γ_3^1 and Γ_3^2

Let the points

$$\vec{X}_1 = \vec{A} + x^1\vec{e}_1 + x^2\vec{e}_2 + x^3\vec{e}_3 \in \Gamma_3^1, \quad \vec{X}_2 = \vec{A} + x^1\vec{e}_1 + x^2\vec{e}_2 + x^4\vec{e}_4 \in \Gamma_3^2 \tag{33}$$

be the current points of the hyperplanes Γ_3^1 and Γ_3^2.

From

$$(d\vec{X}_1, \vec{e}_1, \vec{e}_2, \vec{e}_3) = 0, \ (d\vec{X}_2, \vec{e}_1, \vec{e}_2, \vec{e}_4) = 0 \tag{34}$$

by virtue of (1),(3),(8),(9) and (27) we obtain

$$x^2\omega^2 + x^3(-C^*\omega^2 + E^*\omega^1) = 0, \quad x^1\omega^1 + x^4(-C\omega^1 + E\omega^2) = 0, \tag{35}$$

respectively. It follows from (35) that the next characteristics are the coresponding one of the corresponding hyperplanes along some lines:

Table 1

for hyperplane Γ_3^1	for hyperplane Γ_3^2
1) $\omega^1 = 0 \quad x^2 = x^3 C^*, \ x^4 = 0 \Leftrightarrow$	1) $\omega^2 = 0 \quad x^1 = x^4 C, \ x^3 = 0 \Leftrightarrow$
$\rho_1 = (\vec{A}, \vec{e}_1, C^*\vec{e}_2 + \vec{e}_3)$	$\rho_2 = (\vec{A}, \vec{e}_2, C\vec{e}_1 + \vec{e}_4)$
2) $\omega^2 = 0 \quad x^3 = 0 \Leftrightarrow$	2) $\omega^1 = 0 \quad x^4 = 0 \Leftrightarrow$
$L_2 = (\vec{A}, \vec{e}_1, \vec{e}_2)$ $(E^* \neq 0)$	$L_2 = (\vec{A}, \vec{e}_1, \vec{e}_2)$ $(E \neq 0)$
3) characteristic	3) charactertctic
element $x^2 = x^3 = 0 \Leftrightarrow$	element $x^1 = x^4 = 0 \Leftrightarrow$
$l_1 = (\vec{A}, \vec{e}_1) = \rho_1 \bigcap L_2$	$l_2 = (\vec{A}, \vec{e}_2) = \rho_2 \bigcap L_2$
4) $E^* = 0 \quad x^2 - C^* x^3 = 0, \ x^4 = 0$	4) $E = 0 \quad x^1 - C x^4 = 0, \ x^3 = 0$
charactertctic element	charactertctic element
When $\omega^2 = 0 \quad \Gamma_3^1 \parallel (\Gamma_3^1)'$	When $\omega^1 = 0 \quad \Gamma_3^2 \parallel (\Gamma_3^2)'$

3.6. The hypercone K_2^0

Let us take up the point with the radius vector

$$\vec{X} = \vec{A} + x^\alpha \vec{e}_\alpha + x^{\hat{\alpha}} \vec{e}_{\hat{\alpha}} \in A_4 \tag{36}$$

We shall put

$$\vec{X}_1 = \vec{A} + x^\alpha \vec{e}_\alpha + x^3 \vec{e}_3 = \mathrm{Pr}_{\Gamma_3^1} \vec{X}, \qquad \vec{X}_2 = \vec{A} + x^\alpha \vec{e}_\alpha + x^4 \vec{e}_4 = \mathrm{Pr}_{\Gamma_3^2} \vec{X} \tag{37}$$

Let the point $\vec{X}$ be such a point that $\vec{X}_1$ and $\vec{X}_2$ describe the characteristics of the hyperplanes Γ_3^1 and Γ_3^2 along the corresponding lines. From (34), we obtain (35)

This system has the non–trivial solutions if and only if x^i satisfy the equation:

$$K_2^0: \ x^1 x^2 - C x^2 x^4 - C^* x^1 x^3 - (EE^* - CC^*) x^3 x^4 = 0. \tag{38}$$

Thus, the totality of all points $\vec{X} \in A_4$, which are satisfied the point $\vec{A} \in S_2$, so that the corresponding points (37) lie inside the corresponding characteristical hyperplanes Γ_3^1 and Γ_3^2, forms a second order hypercone K_2^0 in A_4 with the vertex at the point $\vec{A}$. This hypercone is defined by equation (38).

It follows from (38) that the plane

$$\Gamma_2 = (\vec{A}, \vec{e}_2 C^* + \vec{e}_3, C\vec{e}_1 + \vec{e}_4) \tag{39}$$

is polary associated with the plane L_2 in K_2^0.

It follows from table 1 and (39) that after fixation (12) ($C = 0, C^* = 0$)

$$\rho_1 = (\vec{A}, \vec{e}_1, \vec{e}_3),\ \rho_2 = (\vec{A}, \vec{e}_2, \vec{e}_4),\ \Gamma_2 = (\vec{A}, \vec{e}_3, \vec{e}_4) \tag{40}$$

Hence,

$$\begin{aligned} l_3 &= (\vec{A}, \vec{e}_3) = \rho_1 \bigcap \Gamma_2, \\ l_4 &= (\vec{A}, \vec{e}_4) = \rho_2 \bigcap \Gamma_2. \end{aligned} \tag{41}$$

Then the plane

$$P_2 = l_3 \bigcup l_4 \tag{42}$$

can be clothings plane of surface S_2 at a point $\vec{A}$:

$$P_2 \bigcap L_2 = \vec{A},\ P_2 \bigcup L_2 = A_4$$

Taking into consideration (38) and $C = 0$ $C^* = 0$, we notice that the hypercone K_2^0 is defined by the equation

$$K_2^0 :\ x^1 x^2 - EE^* x^3 x^4 = 0. \tag{43}$$

THEOREM 2. *The surface S_2 in A_4 of a class $E = 0$ (or $E^* = 0$) is characterized by that a hypercone K_2^0 is degenerated into two hyperplanes*

$$\begin{aligned} L_3^1 &= (\vec{A}, \vec{e}_2, \vec{e}_3, \vec{e}_4) = l_2 \bigcup \Gamma_2, \\ L_3^2 &= (\vec{A}, \vec{e}_1, \vec{e}_3, \vec{e}_4) = l_2 \bigcup \Gamma_2. \end{aligned} \tag{44}$$

Proof. From the equations (43), (41) and (42) the validity of the equation (44) follows.

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