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NONLINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS WITH SUBDIFFERENTIAL OPERATOR

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1. Introduction

Let H and V be two real Hilbert spaces such that the the corresponding injections $V \subset H \subset V^*$ are densely continuous. Here V^* stands for the dual space of V. Let the operator A be given a single valued operator, which is hemicontinuous and coercive from V to V^*. Let $\phi : V \to (-\infty, +\infty]$ be a lower semicontinuous, proper convex function Then the subdifferential operator $\partial\phi : V \to V^*$ of ϕ is defined by

$$\partial\phi(x) = \{x^* \in V^*; \phi(x) \le \phi(y) + (x^*, x - y), \quad y \in V\}$$

where $(\cdot,\cdot)$ denotes the duality pairing between V^* and V. We are interested in the following nonlinear functional differential equation on H

$$(1.1) \quad \begin{cases} \dfrac{dx(t)}{dt} + Ax(t) + \partial\phi(x(t)) \ni f(t, x(t)) + h(t), \quad 0 < t \le T, \\ x(0) = x_0 \end{cases}$$

where the nonlinear term is given by

$$f(t,x) = \int_0^t k(t-s)g(s,x(s))ds$$

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Here, the nonlinear mapping g is a Lipschitz continuous from $\mathcal{R} \times V$ into H. If A is a linear continuous symmetric operator from V into V^* and satisfies the coercive condition, then the equation (1.1), which is called the linear parabolic variational inequality, was widely developed as seen in section 4.3.2 of Barbu [5]. Using more general hypotheses for nonlinear term $f(\cdot, x)$, we intend to investigate the existence and the norm estimate of a solution of the above nonlinear equation on $L^2(0,T;V) \cap W^{1,2}(0,T;V^*)$, which is also applicable to optimal control problem.

2. Perturbation of subdifferential operator

A norm on V(resp. H) will be denoted by $||\cdot||$ (resp. $|\cdot|$) respectively. The duality pairing between the element v_1 of V^* and the element v_2 of V is denoted by (v_1, v_2), which is the ordinary inner product in H if $v_1, v_2 \in H$. For the sake of simplicity, we may consider

$$||u|| \le |u| \le ||u||_*, \quad u \in V$$

where $||\cdot||_*$ is the norm of the element of V^*.

REMARK 1. If an operator A_0 is bounded linear from V to V^* and generates an analytic semigroup, then it is easily seen that

$$H = \{x \in V^* : \int_0^T ||A_0 e^{tA_0} x||_*^2 dt < \infty\},$$

for the time $T > 0$. Therefore, in terms of the intermediate theory we can see that

$$(V, V^*)_{\frac{1}{2},2} = H$$

where $(V, V^*)_{\frac{1}{2},2}$ denotes the real interpolation space between V and V^*.

We note that a nonlinear operator A is said to be hemicontinuous on V if

$$w - \lim_{t \to 0} A(x + ty) = Ax$$

for every x, $y \in V$ where "$w - \lim$" indicates the weak convergence on V. Let $A : V \to V^*$ be given a single valued and hemicontinuous operator from V to V^* such that

$$\text{(A1)} \qquad A(0) = 0,$$

$$(Au - Av, u - v) \geq \omega_1 ||u - v||^2 - \omega_2 |u - v|^2,$$

$$\text{(A2)} \qquad ||Au||_* \leq \omega_3(||u|| + 1)$$

for every $u, v \in V$ where $\omega_2 \in \mathcal{R}$ and ω_1, ω_3 are some positive constants. Here, we note that if $0 \neq A(0)$ we need the following assumption

$$(Au, u) \geq \omega_1 ||u||^2 - \omega_2 |u|^2$$

for every $u \in V$. It is also known that $A + \omega_2 I$ is maximal monotone and $R(A + \omega_2 I) = V^*$ where $R(A + \omega_2 I)$ is the range of $A + \omega_2 I$ and I is the identity operator.

First, let us concern with the following perturbation of subdifferential operator:

$$\text{(2.1)} \qquad \begin{cases} \dfrac{dx(t)}{dt} + Ax(t) + \partial\phi(x(t)) \ni h(t), \quad 0 < t \leq T, \\ x(0) = x_0 \end{cases}$$

To prove the regularity for nonlinear equation (1.1) without nonlinear term $f(\cdot, x)$ we apply the method of the section 4.3.2 in [5].

THEOREM 2.1. *Let $h \in L^2(0, T; V^*)$ and $x_0 \in V$ satisfying that $\phi(x_0) < \infty$. Then the equation (2.1) has a unique solution*

$$x \in L^2(0, T; V) \cap C([0, T]; H),$$

which satisfies

$$\text{(2.2)} \qquad ||x||_{L^2 \cap C} \leq C_1(1 + ||x_0|| + ||h||_{L^2(0,T;V^*)}).$$

where C_1 is a constant and $L^2 \cap C = L^2(0, T; V) \cap C([0, T]; H)$.

Proof. Substituting $v(t) = e^{\omega_2 t}x(t)$ we can rewrite the equation (2.1) as follows:

$$
\begin{cases} \dfrac{dv(t)}{dt} + (A+\omega_2 I)v(t) + e^{-\omega_2 t}\partial\phi(v(t)) \ni e^{-\omega_2 t}h(t), \\ \hfill 0 < t \le T, \\ v(0) = e^{\omega_2 t}x_0. \end{cases} \tag{2.3}
$$

Then the regular problem for the equation (2.1) is equivalent to that for (2.3). Consider the operator $L : D(L) \subset H \to H$

$$
\begin{aligned} Lv &= \{Av + e^{-\omega_2 t}\partial\phi(v) + \omega_2 v\} \cap H, \quad \forall v \in D(L) \\ D(L) &= \{v \in V; \{Av + e^{-\omega_2 t}\partial\phi(v) + \omega_2 v\} \cap H \neq \emptyset\}. \end{aligned}
$$

Since $A + \omega_2 I$ is a monotone, hemicontinuous and bounded operator from V into V^* and $e^{-\omega_2 t}\partial\phi$ is a maximal monotone, we infer by Corollary 1.1 of Chapter 2 in [4] that L is maximal monotone. Then by Theorem 1.4 in [5](also see Theorem 2.3 and Corollary 2.1 in [4]), for every $x_0 \in D(L)$ and $h \in W^{1,1}([0,T];H)$ the Cauchy problem (2.3) has a unique solution $v \in W^{1,\infty}([0,T];H)$. Let us assume that $x_0 \in D(L)$ and $h \in W^{1,2}(0,T;H)$. Multiplying (2.1) by $x - x_0$ and using (A1) and the maximal monotonicity of $\partial\phi$ it holds

$$
\begin{aligned} \frac{1}{2}\frac{d}{dt}|x(t)-x_0|^2 + \omega_1||x(t)-x_0||^2 &\le \omega_2|x(t)-x_0| \\ &+ (h(t) - Ax_0 - \partial\phi(x_0), x(t)-x_0). \end{aligned} \tag{2.4}
$$

Since

$$
\begin{aligned} &(h(t) - Ax_0 - \partial\phi(x_0), x(t)-x_0) \\ &\le ||h(t) - Ax_0 - \partial\phi(x_0)||_* ||x(t)-x_0)|| \\ &\le \frac{1}{2c^2}||h(t) - Ax_0 - \partial\phi(x_0)||_*^2 + \frac{c^2}{2}||x(t)-x_0||^2 \end{aligned}
$$

for every real number c, so using Gronwall's Lemma, inequality (2.4) implies that

$$
|x(t)-x_0|^2 + \int_0^t ||x(s)-x_0||^2 ds \le C_1\Big(\int_0^t ||h(s)||_*^2 ds + ||x_0||^2 + 1\Big)
$$

for some positive constant C_1, that is,

$$(2.5) \qquad ||x||_{L^2(0,T,V)\cap C([0,T],H)} \le C_1(1 + ||x_0|| + ||h||_{L^2(0,T,V^*)}).$$

Hence we have proved (2.2). Let $x_0 \in V$ satisfying $\partial\phi(x_0) < \infty$ and $h \in L^2(0,T;V^*)$. Then there exist sequences $\{x_{0n}\} \subset D(L)$ and $\{h_n\} \subset W^{1,2}(0,T;H)$ such that $x_{0n} \to x_0$ in V and $h_n \to h$ in $L^2(0,T;V^*)$ as $n \to \infty$. Let $x_n \in W^{1,\infty}(0,T;H)$ be the solution of (2.1) with the initial value x_{0n} and with h_n instead of h. Since $\partial\phi$ is monotone, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|x_n(t) - x_m(t)|^2 + \omega_1||x_n(t) - x_m(t)||^2 \\
&\le (h_n(t) - h_m(t), x_n(t) - x_m(t)) + \omega_2|x_n(t) - x_m(t)|^2 \\
&\le \frac{1}{2c^2}||h_n(t) - h_m(t)||_*^2 + \frac{c^2}{2}||x_n(t) - x_m(t)||^2 \\
&\qquad + \omega_2|x_n(t) - x_m(t)|^2, \quad \text{a.e. }, t \in (0,T),
\end{aligned}$$

for every real number c. Therefore, if we choose c so that $\omega_1 - c^2/2 > 0$ then by integrating over $[0,T]$ and using Gronwall's inequality it follows that

$$\begin{aligned}
&|x_n(t) - x_m(t)| + 2(\omega_1 - \frac{c^2}{2})||x_n(t) - x_m(t)||_{L^2(0,T,V)} \\
&\le e^{2\omega_2 T_1}(|x_{0n} - x_{0m}| + c^{-2}||h_n - h_m||_{L^2(0,T,V^*)}),
\end{aligned}$$

and hence, we have that $\lim_{n\to\infty} x_n(t) = x(t)$ exists in H. Furthermore, by using the maximal monotonicity of $A + \partial\phi + \omega_2 I$, it is easily seen that x satisfies (2.1).

3. Nonlinear integrodifferential equation

Let $g : [0,T] \times V \to H$ be a nonlinear mapping satisfying the following:

(g1) $\qquad |g(t,x) - g(t,y)| \le L||x - y||$

(g2) $\qquad g(t,0) = 0$

for a positive constant L.

For $x \in L^2(0,T;V)$ we set

$$f(t,x) = \int_0^t k(t-s)g(s,x(s))ds$$

where k belongs to $L^2(0,T)$.

REMARK 2. If $g : [0,T] \times H \to H$ is a nonlinear mapping satisfying

$$|g(t,x) - g(t,y)| \le L|x-y|$$

for a positive constant L, then as is seen in [1], our results can be obtained directly.

LEMMA 3.1. *Let* $x \in L^2(0,T;V)$, $T > 0$. *Then* $f(\cdot,x) \in L^2(0,T;H)$ *and*

$$||f(\cdot,x)||_{L^2(0,T;H)} \le L||k||_{L^2(0,T)}\sqrt{T}||x||_{L^2(0,T;V)}.$$

Moreover if $x_1,\ x_2 \in L^2(0,T;V)$, *then*

$$||f(\cdot,x_1) - f(\cdot,x_2)||_{L^2(0,T;H)} \le L||k||\sqrt{T}||x_1 - x_2||_{L^2(0,T,V)}.$$

Proof. From (g1), (g2) and using the Hölder inequality it is easily seen that

$$\begin{aligned} ||f(\cdot,x)||^2_{L^2(0,T;H)} &\le \int_0^T |\int_0^t k(t-s)g(s,x(s))ds|^2 dt \\ &\le ||k||^2_{L^2} \int_0^T \int_0^t L^2 ||x(s)||^2 dsdt \\ &\le TL^2||k||^2_{L^2}||x||^2_{L^2(0,T;V)}. \end{aligned}$$

The proof of the second paragraph is similar.

THEOREM 3.1. *Let (A1), (A2), (g1) and (g2) be satisfied. Then (1.1) has a unique solution*

$$x \in L^2(0,T;V) \cap C([0,T];H).$$

Furthermore, there exists a constant C_2 such that

$$||x||_{L^2 \cap C} \le C_2(1 + ||x_0|| + ||h||_{L^2(0,T;V^*)}). \tag{3.1}$$

If $(x_0, h) \in V \times L^2(0,T;V^*)$, then $x \in L^2(0,T;V) \cap C([0,T];H)$ and the mapping

$$V \times L^2(0,T;V^*) \ni (x_0, h) \mapsto x \in L^2(0,T;V) \cap C([0,T];H)$$

is continuous.

Proof. Let $y \in L^2(0,T;V)$. Then $f(\cdot, y(\cdot)) \in L^2(0,T,H)$ from Lemma 3.1. Thus, in virtue of Proposition 2.1 we know that the problem

$$\begin{cases} \dfrac{dx(t)}{dt} + Ax(t) + \partial\phi(x(t)) \ni f(t, y(t)) + h(t), \quad 0 < t \le T, \\ x(0) = x_0 \end{cases} \tag{3.2}$$

has a unique solution $x_y \in L^2(0,T;V) \cap C([0,T],H)$ corresponding To y.

let us choose a constant $c > 0$ such that

$$\omega_1 - c^2/2 > 0$$

and let us fix $T_0 > 0$ so that

$$(2c^2\omega_1 - c^4)^{-1} e^{2\omega_2 T_0} L||k||\sqrt{T}_0 < 1 \tag{3.3}$$

Let x_i, $i = 1, 2$, be solutions of (3.2) corresponding to y_i. Then, by the monotonicity of $\partial\phi$, it follows that

$$\begin{aligned} &(\dot{x}_1(t) - \dot{x}_2(t), x_1(t) - x_2(t)) + (Ax_1(t) - Ax_2(t), x_1(t) - x_2(t)) \\ &\le (f(t, y_1(t)) - f(t, y_2(t)), x_1(t) - x_2(t)), \end{aligned}$$

and hence, using the assumption (A1), we have that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|x_1(t)-x_2(t)|^2+\omega_1||x_1(t)-x_2(t)||^2 \\ (3.4)\qquad &\le \omega_2|x_1(t)-x_2(t)|^2 \\ &\qquad + ||f(t,y_1(t))-f(t,y_2(t))||_*||x_1(t)-x_2(t)||. \end{aligned}$$

Since

$$\begin{aligned} &||f(t,y_1(t))-f(t,y_2(t))||_*||x_1(t)-x_2(t)|| \\ &\le \frac{1}{2c^2}||f(t,y_1(t))-f(t,y_2(t))||_*^2+\frac{c^2}{2}||x_1(t)-x_2(t)||^2 \end{aligned}$$

for every $c > 0$ and by integrating on (3.4) over $(0, T_0)$ we have

$$\begin{aligned} &|x_1(T_0)-x_2(T_0)|^2+(2\omega_1-c^2)\int_0^{T_0}||x_1(t)-x_2(t)||^2dt \\ &\le \frac{1}{c^2}||f(t,y_1)-f(t,y_2)||_{L^2(0,T_0;V^*)}+2\omega_2\int_0^{T_0}|x_1(t)-x_2(t)|^2dt \end{aligned}$$

and by Gronwall's inequality,

$$||x_1-x_2||^2_{L^2(0,T_0,V)} \le (2c^2\omega_1-c^4)^{-1}e^{2\omega_2 T_0}||f(t,y_1)-f(t,y_2)||^2_{L^2(0,T_0,V^*)}.$$

Thus, from (g1) it follows that

$$||x_1-x_2||_{L^2} \le (2c^2\omega_1-c^4)^{-1}e^{2\omega_2 T_0}L||k||\sqrt{T}_0||y_1-y_2||_{L^2(0,T_0,V)}.$$

Hence we have proved that $y \mapsto x$ is strictly contraction from $L^2(0, T_0; V)$ to itself if the condition (3.3) is satisfied. It gives the equation (1.1) has a unique solution in $[0, T_0]$.

Let y be the solution of

$$\begin{cases} \dfrac{dy(t)}{dt}+Ay(t)+\partial\phi(y(t)) \ni 0, \quad 0 < t \le T_0, \\ y(0)=x_0. \end{cases}$$

Then, since

$$\frac{d}{dt}(x(t)-y(t))+(Ax(t)-Ay(t))+(\partial\phi(x(t))-\partial\phi(y(t))) \ni f(t,x(t))+h(t),$$

by multiplying by $x(t)-y(t)$ and using the monotonicity of $\partial\phi$, we obtain

$$\begin{aligned}&\cdot\frac{1}{2}\frac{d}{dt}|x(t)-y(t)|^2+\omega_1||x(t)-y(t)||^2\\ &\leq \omega_2|x(t)-y(t)|^2+||f(t,x(t))+h(t)||_*||x(t)-y(t)||.\end{aligned}$$

Therefore, putting

$$N=(2c^2\omega_1-c^4)^{-1}e^{2\omega_2 T_0},$$

from (g1) it follows

$$\begin{aligned}||x-y||_{L^2(0,T_0,V)} &\leq N||f(\cdot,x)+h||_{L^2(0,T_0,V^*)}\\ &\leq NL||k||\sqrt{T}_0||x||_{L^2(0,T_0,V)}+N||h||_{L^2(0,T_0\,V^*)}\end{aligned}$$

and hence

$$\begin{aligned}&||x||_{L^2(0,T_0;V)}\\ &\leq \frac{1}{1-NL||k||\sqrt{T_0}}||y||_{L^2(0,T_0,V)}\\ &\leq \frac{C_1}{1-NL||k||\sqrt{T_0}}(1+||x_0||+||h||_{L^2(0,T_0,V^*)})\\ &\leq C_2(1+||x_0||+||h||_{L^2(0,T_0,V^*)})\end{aligned} \tag{3.5}$$

for some positive constant C_2. Since the condition (3.3) is independent of initial values, the solution of (1.1) can be extended the internal $[0,nT_0]$ for natural number n, i.e., for the initial $x(nT_0)$ in the interval $[nT_0,(n+1)T_0]$, as analogous estimate (3.5) holds for the solution in $[0,(n+1)T_0]$. Furthermore, by the similar way as (2.4) and (2.5) in section 2, the estimate (3.1) is easily obtained.

Now we prove the last paragraph. If $(x_0,h)\in V\times L^2(0,T;V^*)$ then x belongs to $L^2(0,T;V)$ Let $(x_{0i},h_i)\in V\times L^2(0,T;V^*)$ and x_i

be the solution of (1.1) with (x_{0i}, h_i) in place of (x_0, u) for $i = 1,\ 2$. Multiplying on (1.1) by $x_1(t) - x_2(t)$, we have

(3.6)

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|x_1(t) - x_2(t)|^2 + \omega_1||x_1(t) - x_2(t)||^2 \\ &\le \omega_2|x_1(t) - x_2(t)|^2 + ||f(t, y_1(t)) - f(t, y_2(t))||_*||x_1(t) - x_2(t)|| \\ &\qquad + ||h_1(t) - h_2(t)||_*||x_1(t) - x_2(t)||\end{aligned}$$

If $\omega_1 - c^2/2 > 0$ then we can choose a constant $c_1 > 0$ so that

$$\omega_1 - c^2/2 - c_1^2/2 > 0$$

and

$$\begin{aligned}||h_1(t) - h_2(t))||_*||x_1(t) - x_2(t)|| \le &\frac{1}{2c_1^2}||h_1(t) - h_2(t)||_*^2 \\ &+ \frac{c_1^2}{2}||x_1(t) - x_2(t)||^2.\end{aligned}$$

Let $T_1 < T$ be such that

$$2\omega_1 - c^2 - c_1^2 - c^{-2}e^{2\omega_2 T_1}L||k||\sqrt{T}_1 > 0.$$

Integrating on (3.6) over $[0, T_1]$ where $T_1 < T$ and as is seen in the first part of proof, it follows

$$\begin{aligned}&(2\omega_1 - c^2 - c_1^2)||x_1 - x_2||^2_{L^2(0,T_0;V)} \\ &\le e^{2\omega_2 T_1}\{||x_{01} - x_{02}|| + \frac{1}{c^2}||f(t, y_1) - f(t, y_2)||^2_{L^2(0,T_0;V^*)} \\ &\qquad + \frac{1}{c_1^2}||h_1 - h_2||_{L^2(0,T,V^*)}\} \\ &\le e^{2\omega_2 T_1}\{||x_{01} - x_{02}|| + \frac{1}{c^2}L||k||\sqrt{T}_1||x_1 - x_2||^2_{L^2(0,T_0,V)} \\ &\qquad + \frac{1}{c_1^2}||h_1 - h_2||_{L^2(0,T;V^*)}\}.\end{aligned}$$

Putting that

$$N_1 = 2\omega_1 - c^2 - c_1^2 - c^{-2}e^{2\omega_2 T_1}L||k||\sqrt{T}_1$$

we have

$$(3.7) \qquad ||x_1 - x_2||_{L^2} \leq \frac{e^{2\omega_2 T_1}}{N_1}(||x_{01} - x_{02}|| + \frac{1}{c_1^2}||h_1 - h_2||).$$

Suppose $(x_{0n}, h_n) \to (x_0, h)$ in $H \times L^2(0, T; V^*)$, and let x_n and x be the solutions (1.1) with (x_{0n}, h_n) and (x_0, h), respectively. Then, by virtue of (3.7) and (3.6), we see that $x_n \to x$ in $L^2(0, T_1, V) \cap C([0, T_1]; H)$. This implies that $x_n(T_1) \to x(T_1)$ in V. Therefore the same argument shows that $x_n \to x$ in

$$L^2(T_1, \min\{2T_1, T\}; V) \cap C([T_1, \min\{2T_1, T\}], H).$$

Repeating this process, we conclude that $x_n \to x$ in $L^2(0, T; V) \cap C([0, T], H)$.

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