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LIMIT THEOREMS ON ONE-SIDED INCREMENTS OF A TWO-PAREMTER FRACTIONAL LÉVY BROWNIAN MOTION

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ABSTRACT. Here we establish limit theorems on one-side increments of two-parameter fractional Lévy Brownian motion, via estimating upper bounds large deviation probability inequalities on the suprema of the fractional Lévy Brownian motion.

1. Introduction and results

Some results on the increments of a Wiener process and a Gaussian process deeply describe the properties of their sample paths. P. Lévy [8] established initial results for the increments of one-parameter Wiener process. Since then, limit theorems on the increments of one-parameter Wiener processes or Gaussian processes with stationary increments were established by Csörgő-Révész [4], Choi [1], Csáki et al [3], Choi et al [2], etc. Furthermore, Csorgő-Révész ([5], pp. 58-87) referred to the Lévy type limit theorems for two-parameter Wiener processes.

In this paper, we establish limit theorems on one-sided increments of two-parameter fractional Lévy Brownian motion: A *two-parameter fractional Lévy Brownian motion* $\{X(x,y), 0 \le x, y < \infty\}$ of order 2α with $0 < \alpha < 1$, on an underlying probability space $(\Omega, \mathcal{A}, P)$, is a real-valued Gaussian process satisfying the following conditions:

(1) (a) $X(x,y)$ is almost surely continuous on $[0,\infty) \times [0,\infty)$,

(2) (b) $X(0,0) = 0$ and $E\{X(x,y)\} = 0$,

(3) (c) $X(x,y)$ has stationary increments: for all distinct two points (x_1,y_1), (x_2,y_2) in $[0,\infty) \times [0,\infty)$, we have

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(4) $E\{X(x_1,y_1)-X(x_2,y_2)\}^2 = \big((x_1-x_2)^2+(y_1-y_2)^2\big)^\alpha$, $0 < \alpha < 1$.

For $0 < T < \infty$, let a_T be a nondecreasing function of T such that
(i) $0 < a_T \le T$,
(ii) T/a_T is nondecreasing.

Thoughout the paper, we always assume that $X(x,y)$ and a_T satisfy the above conditions. Our main results are as follows:

THEOREM 1.1. *We have*

$$\limsup_{T\to\infty} \sup_{0\le s\le a_T} \sup_{0\le t\le T-s} \sup_{0\le v\le T} \frac{|X(t+s,v)-X(t,v)|}{(a_T)^\alpha \gamma_T} \le 1 \qquad \text{a.s.},$$

where $\gamma_T = 2\big\{\log(T/a_T) + \frac{1}{2}\log\log T\big\}^{1/2}$.

THEOREM 1.2. *If, in addition, the following condition is also satisfied:*
(iii) $\lim_{T\to\infty} \log(T/a_T)\big/\log\log T = \infty$,

then we have

$$\liminf_{T\to\infty} \sup_{0\le t\le T-a_T} \sup_{0\le v\le T} \frac{|X(t+a_T,v)-X(t,v)|}{(a_T)^\alpha \gamma_T} \ge 1 \qquad \text{a.s.}$$

From Theorems 1.1 and 1.2, we immediately obtain the following limit theorem:

COROLLARY 1.1. *Under the assumptions of Theorem 1.2, we have*

$$\lim_{T\to\infty} \sup_{0\le s\le a_T} \sup_{0\le t\le T-s} \sup_{0\le v\le T} \frac{|X(t+s,v)-X(t,v)|}{(a_T)^\alpha \gamma_T} = 1 \qquad \text{a.s.}$$

and

$$\lim_{T\to\infty} \sup_{0\le t\le T-a_T} \sup_{0\le v\le T} \frac{|X(t+a_T,v)-X(t,v)|}{(a_T)^\alpha \gamma_T} = 1 \qquad \text{a.s.}$$

EXAMPLE 1.1. (1) Set $a_T = (\log T)^{1/2}$ in Corollary 1.1. Then, for $0 < \alpha < 1$,

$$\lim_{T\to\infty} \sup_{0\le s\le (\log T)^{1/2}} \sup_{0\le t\le T-s} \sup_{0\le v\le T} \frac{|X(t+s,v) - X(t,v)|}{2(\log T)^{(\alpha+1)/2}} = 1 \qquad \text{a.s.}$$

(2) Set $a_T = 1$ in Corollary 1.1. Then

$$\lim_{T\to\infty} \sup_{0\le t\le T-1} \sup_{0\le v\le T} \frac{|X(t+1,v) - X(t,v)|}{2(\log T)^{1/2}} = 1 \qquad \text{a.s.}$$

2. Proofs

To prove the theorems, we first introduce the following Lemma 2.1, due to Fernique [6]: Let $\mathbb{D} = \{\mathbf{t} : \mathbf{t} = (t_1, t_2, \cdots, t_d),\ a_i \le t_i \le b_i,\ i = 1, 2, \cdots, d\}$ be a real d-dimensional time parameter space with the usual Euclidean norm $\|\cdot\|$. Let $\{X(\mathbf{t}),\ \mathbf{t} \in \mathbb{D}\}$ be a real-valued separable Gaussian process with $E\{X(\mathbf{t})\} = 0$. Suppose that

$$0 < \sup_{\mathbf{t}\in\mathbb{D}} E\{X(\mathbf{t})\}^2 =: \Gamma^2 < \infty, \quad \Gamma > 0,$$

and

$$E\{X(\mathbf{t}) - X(\mathbf{s})\}^2 \le \varphi^2(\|\mathbf{t} - \mathbf{s}\|),$$

where $\varphi(\cdot)$ is a nondecreasing continuous function such that $\int_0^\infty \varphi(e^{-y^2})\,dy < \infty$.

LEMMA 2.1. *Let $\{X(\mathbf{t}),\ \mathbf{t} \in \mathbb{D}\}$ be given as above statements. Then, for $\lambda > 0$, $x \ge 1$ and $A > 2\sqrt{d\log 2}$, we have*

$$P\Big\{\sup_{\mathbf{t}\in\mathbb{D}} X(\mathbf{t}) > x\Big(\Gamma + (2\sqrt{2}+2)A\int_0^\infty \varphi(\sqrt{d}\lambda 2^{-y^2})\,dy\Big)\Big\}$$

$$\le (2^{2d} + \psi)\Big(\prod_{i=1}^{d}\Big(\frac{b_i - a_i}{\lambda} \vee 1\Big)\Big)e^{-x^2/2},$$

where $x \vee y = \max\{x, y\}$ and $\psi = \sum_{n=1}^\infty \exp\Big\{\frac{1}{2} - 2^n\Big(\frac{A^2}{2} - 2d\log 2\Big)\Big\} < \infty$.

Let us estimate an upper bound of the following large deviation probabilities by using Lemma 2.1.

LEMMA 2.2. *For any $\varepsilon > 0$ there exists a positive constant C_ε depending only on ε such that*

$$P\Big\{\sup_{0\le s\le a_T}\sup_{0\le t\le T-s}\sup_{0\le v\le T}\frac{|X(t+s,v)-X(t,v)|}{(a_T)^\alpha}\ge u\Big\}$$
$$\le C_\varepsilon\Big(\frac{T}{a_T}\Big)^2 e^{-u^2/(2+\varepsilon)}$$

for all $u > 0$.

Proof. Let $\mathbb{A} = \{(t,s,v) : 0\le t\le T-s,\ 0\le s\le a_T,\ 0\le v\le T\}$ be a three-dimensional space. In order to apply Lemma 2.1, we set

$$Y(t,s,v) = \frac{X(t+s,v)-X(t,v)}{(a_T)^\alpha}, \quad (t,s,v)\in\mathbb{A},$$

and

$$\varphi(z) = \frac{2(\sqrt{2}z)^\alpha}{(a_T)^\alpha}, \quad z>0.$$

Clearly, $E\{Y(t,s,v)\} = 0$ for all $(t,s,v)\in\mathbb{A}$ and $\Gamma^2 = 1$. Using the elementary inequality $(a\pm b)^2 \le 2(a^2+b^2)$, we have

$$\begin{aligned}
&E\{Y(t_1,s_1,v_1)-Y(t_2,s_2,v_2)\}^2\\
&\le \frac{2}{(a_T)^{2\alpha}}\big(E\{X(t_1+s_1,v_1)-X(t_2+s_2,v_2)\}^2\\
&\qquad + E\{X(t_1,v_1)-X(t_2,v_2)\}^2\big)\\
&\le \frac{4}{(a_T)^{2\alpha}}\big(\sqrt{2}\sqrt{(t_1-t_2)^2+(s_1-s_2)^2+(v_1-v_2)^2}\,\big)^{2\alpha}
\end{aligned}$$

for all $(t_1,s_1,v_1),(t_2,s_2,v_2)$ in $\mathbb{A}$ Letting $\mathbf{u} = (t_1,s_1,v_1)$ and $\mathbf{v} = (t_2,s_2,v_2)$ in $\mathbb{A}$, it follows that

$$E\{Y(\mathbf{u})-Y(\mathbf{v})\}^2 \le \varphi^2(\|\mathbf{u}-\mathbf{v}\|).$$

For any $\varepsilon>0$ there exists a small constant $c>0$ such that

$$(2\sqrt{2}+2)A\int_0^\infty \varphi(\sqrt{3}\,c\,a_T\,2^{-y^2})\,dy < \frac{\varepsilon}{8},$$

where A is a constant such that $A > 2\sqrt{3\log 2}$. Indeed, for any $\varepsilon > 0$ there exists a small $c = c(\varepsilon) > 0$ such that

$$\int_0^\infty \varphi(\sqrt{3}\, c\, a_T\, 2^{-y^2})\, dy = 2(\sqrt{6}\, c)^\alpha \int_0^\infty 2^{-\alpha y^2}\, dy$$
$$= (\sqrt{6}c)^\alpha \sqrt{\frac{\pi}{\alpha \log 2}} < (\varepsilon/8)/((2\sqrt{2}+2)A).$$

Let $u = x(1 + (\varepsilon/8))$, $x \geq 1$. Then it follows from Lemma 2.1 that

$$P\Big\{\sup_{(t,s,v)\in \mathrm{A}} |Y(t,s,v)| \geq u\Big\}$$
$$\leq 2P\Big\{\sup_{(t,s,v)\in \mathrm{A}} Y(t,s,v) \geq x\Big(1 + (2\sqrt{2}+2)A \int_0^\infty \varphi(\sqrt{3}\, c\, a_T\, 2^{-y^2})\, dy\Big)\Big\}$$
$$\leq 2(4^3 + \psi)\Big(\frac{a_T}{ca_T} \vee 1\Big)\Big(\frac{T-s}{ca_T} \vee 1\Big)\Big(\frac{T}{ca_T} \vee 1\Big)e^{-x^2/2}$$
$$\leq C_\varepsilon \Big(\frac{T}{a_T}\Big)^2 e^{-u^2/(2+\varepsilon)},$$

where C_ε is a positive constant depending only on $\varepsilon > 0$. In case $0 < u \leq 1$, the result is trivial if we take C_ε large enough. □

Proof of Theorem 1.1. Put $T_k = \exp(k^\beta), k \in \mathbb{N}$, where $\frac{2+\varepsilon}{2+2\varepsilon} < \beta < 1$, $\varepsilon > 0$, and $\mathbb{N}$ is the set of positive integers. Let T be in $T_{k-1} \leq T \leq T_k$. It follows from the condition (ii) that $T - a_T \leq T_k - a_{T_k}$, and hence the following inequalities hold:

(2 1)
$$\sup_{0\leq s\leq a_T}\ \sup_{0\leq t\leq T-s}\ \sup_{0\leq v\leq T} \frac{|X(t+s,v) - X(t,v)|}{(a_T)^\alpha \gamma_T}$$
$$\leq \sup_{0\leq s\leq a_{T_k}}\ \sup_{0\leq t\leq T_k-s}\ \sup_{0\leq v\leq T_k} \frac{|X(t+s,v) - X(t,v)|}{2(a_{T_k})^\alpha \sqrt{\log(T_k/a_{T_k}) + \frac{1}{2}\log\log T_k}}$$
$$\times \frac{(a_{T_k})^\alpha}{(a_{T_{k-1}})^\alpha} \sqrt{\frac{\log(T_k/a_{T_k}) + \frac{1}{2}\log\log T_k}{\log(T_{k-1}/a_{T_{k-1}}) + \frac{1}{2}\log\log T_{k-1}}}$$

By (ii) and the mean-value theorem,

$$1 \le \frac{a_{T_k}}{a_{T_{k-1}}} \le \frac{T_k}{T_{k-1}} \le \exp(\beta(k-1)^{\beta-1}) \longrightarrow 1 \quad \text{as} \quad k \to \infty, \tag{2.2}$$

$$\begin{aligned} 1 &\le \frac{\log(T_k/a_{T_k}) + \frac{1}{2}\log\log T_k}{\log(T_{k-1}/a_{T_{k-1}}) + \frac{1}{2}\log\log T_{k-1}} \\ &\le \frac{(T_k/a_{T_k})\sqrt{\log T_k}}{(T_{k-1}/a_{T_{k-1}})\sqrt{\log T_{k-1}}} \\ &\longrightarrow 1 \quad \text{as} \quad k \to \infty. \end{aligned} \tag{2.3}$$

Combining (2.1), (2.2) and (2.3), we get

$$\begin{aligned} &\limsup_{T\to\infty} \sup_{0\le s\le a_T} \sup_{0\le t\le T-s} \sup_{0\le v\le T} \frac{|X(t+s,v)-X(t,v)|}{(a_T)^\alpha \gamma_T} \\ &\le \limsup_{k\to\infty} \sup_{0\le s\le a_{T_k}} \sup_{0\le t\le T_k-s} \sup_{0\le v\le T_k} \frac{|X(t+s,v)-X(t,v)|}{(a_{T_k})^\alpha \gamma_{T_k}}. \end{aligned} \tag{2.4}$$

On the other hand, it follows from Lemma 2.2 that for any $\varepsilon > 0$ there exists a positive constant C_ε such that

$$\begin{aligned} &P\Big\{\sup_{0\le s\le a_{T_k}} \sup_{0\le t\le T_k-s} \sup_{0\le v\le T_k} \frac{|X(t+s,v)-X(t,v)|}{(a_{T_k})^\alpha \gamma_{T_k}} > \sqrt{1+\varepsilon}\Big\} \\ &\le C_\varepsilon\Big(\frac{T_k}{a_{T_k}}\Big)^2 \exp\Big\{-\frac{4+4\varepsilon}{2+\varepsilon}\log\Big(\frac{T_k}{a_{T_k}}\sqrt{\log T_k}\Big)\Big\} \\ &= C_\varepsilon\Big(\frac{T_k}{a_{T_k}}\Big)^{-2\varepsilon/(2+\varepsilon)} (\log T_k)^{-1-(\varepsilon/(2+\varepsilon))} \\ &\le C_\varepsilon k^{-\beta(2+2\varepsilon)/(2+\varepsilon)}. \end{aligned}$$

Hence the series

$$P\Big\{\sup_{0\le s\le a_{T_k}} \sup_{0\le t\le T_k-s} \sup_{0\le v\le T_k} \frac{|X(t+s,v)-X(t,v)|}{(a_{T_k})^\alpha \gamma_{T_k}} > \sqrt{1+\varepsilon}\Big\}$$

is convergent and

$$\limsup_{k\to\infty} \sup_{0\le s\le a_{T_k}} \sup_{0\le t\le T_k - s} \sup_{0\le v\le T_k} \frac{|X(t+s,v)-X(t,v)|}{(a_{T_k})^\alpha \gamma_{T_k}} \le 1 \quad \text{a.s.} \tag{2.5}$$

Combining (2.4) and (2.5), we obtain Theorem 1.1. □

For proving Theorem 1.2, we need the following Lemmas 2.3~2.6:

LEMMA 2.3. *Let $a > 0$ and let N, l be positive integers and $q(\ne 0)$ a real number. Then there exists a constant $C > 0$ such that*

$$\begin{aligned}&\left|\int_{\sqrt{q^2+(Nla)^2}}^{\sqrt{q^2+(Nla+a)^2}} d(x^{2\alpha}) - \int_{\sqrt{q^2+(Nla-a)^2}}^{\sqrt{q^2+(Nla)^2}} d(x^{2\alpha})\right| \\ &\le C\frac{\left(q^2+(Nla+a)^2\right)^\alpha a^2}{q^2+(Nla-a)^2}.\end{aligned} \tag{2.6}$$

Proof. Set $b = Nla - a$, $c = Nla$ and $d = Nla + a$. Then

$$\begin{aligned}&\int_{\sqrt{q^2+c^2}}^{\sqrt{q^2+d^2}} d(x^{2\alpha}) - \int_{\sqrt{q^2+b^2}}^{\sqrt{q^2+c^2}} d(x^{2\alpha}) \\ &= \int_{\sqrt{q^2+b^2}}^{\sqrt{q^2+d^2}+\sqrt{q^2+b^2}-\sqrt{q^2+c^2}} d\big((z+\sqrt{q^2+c^2}-\sqrt{q^2+b^2}\,)^{2\alpha}\big) \\ &\quad - \int_{\sqrt{q^2+b^2}}^{\sqrt{q^2+c^2}} d(x^{2\alpha}) \\ &= \int_{\sqrt{q^2+b^2}}^{\sqrt{q^2+d^2}+\sqrt{q^2+b^2}-\sqrt{q^2+c^2}} \Big(\frac{d\big((x+\sqrt{q^2+c^2}-\sqrt{q^2+b^2}\,)^{2\alpha}\big)}{dx} - \frac{d(x^{2\alpha})}{dx}\Big) \\ &\quad + \int_{\sqrt{q^2+c^2}}^{\sqrt{q^2+d^2}+\sqrt{q^2+b^2}-\sqrt{q^2+c^2}} \frac{d(x^{2\alpha})}{dx}\, dx\end{aligned}$$

$$= \int_{\sqrt{q^2+b^2}}^{\sqrt{q^2+d^2}+\sqrt{q^2+b^2}-\sqrt{q^2+c^2}} \Big(\int_x^{x+\sqrt{q^2+c^2}-\sqrt{q^2+b^2}} \frac{d^2(z^{2\alpha})}{dz^2}\, dz \Big)\, dx$$

$$+ \int_{\sqrt{q^2+c^2}}^{\sqrt{q^2+d^2}+\sqrt{q^2+b^2}-\sqrt{q^2+c^2}} \frac{d(x^{2\alpha})}{dx}\, dx$$

$$= \mathrm{I} + \mathrm{J}, \quad \text{say}.$$

Let us estimate an upper bound for I. Then for some $C_2 > 0$

$$|\mathrm{I}|$$

$$\le \int_{\sqrt{q^2+b^2}}^{\sqrt{q^2+d^2}+\sqrt{q^2+b^2}-\sqrt{q^2+c^2}} \Big(\int_x^{x+\sqrt{q^2+c^2}-\sqrt{q^2+b^2}} |2\alpha(2\alpha-1)| \frac{z^{2\alpha}}{z^2}\, dz \Big)\, dx$$

$$\le C_2 \int_{\sqrt{q^2+b^2}}^{\sqrt{q^2+d^2}+\sqrt{q^2+b^2}-\sqrt{q^2+c^2}} \frac{(x+\sqrt{q^2+c^2}-\sqrt{q^2+b^2})^{2\alpha}}{x^2}$$

$$\times \big(\sqrt{q^2+c^2}-\sqrt{q^2+b^2}\big)\, dx$$

$$\le C_2 \frac{(q^2+d^2)^\alpha}{q^2+b^2} \big(\sqrt{q^2+d^2}-\sqrt{q^2+c^2}\big)\big(\sqrt{q^2+c^2}-\sqrt{q^2+b^2}\big)$$

$$\le C_2 \frac{(q^2+d^2)^\alpha}{q^2+b^2}(d-c)(c-b) = C_2 \frac{(q^2+(Nla+a)^2)^\alpha a^2}{q^2+(Nla-a)^2}.$$

As for J, we have

$$|\mathrm{J}| = 2\alpha \int_{\sqrt{q^2+c^2}}^{\sqrt{q^2+d^2}+\sqrt{q^2+b^2}-\sqrt{q^2+c^2}} \frac{x^{2\alpha}}{x}\, dx$$

$$\le 2\frac{(q^2+d^2)^\alpha}{\sqrt{q^2+c^2}} \big(\sqrt{q^2+d^2}-\sqrt{q^2+c^2}-\big(\sqrt{q^2+c^2}-\sqrt{q^2+b^2}\big)\big)$$

$$= 2\frac{(q^2+d^2)^\alpha}{\sqrt{q^2+c^2}} \Big(\frac{2Nla^2+a^2}{\sqrt{q^2+d^2}+\sqrt{q^2+c^2}} - \frac{2Nla^2-a^2}{\sqrt{q^2+c^2}+\sqrt{q^2+b^2}} \Big)$$

$$\le 2\frac{(q^2+d^2)^\alpha}{\sqrt{q^2+c^2}} \frac{2a^2}{\sqrt{q^2+c^2}} = 4\frac{(q^2+(Nla+a)^2)^\alpha a^2}{q^2+(Nla)^2}. \qquad \Box$$

The following Lemma 2.4 is a modification of Theorem 4.2.1 in Leadbetter et al. ([7], pp. 81-84):

LEMMA 2.4. *Let $\{Y_{ij}, i, j = 1, 2, \ldots, n\}$ be jointly standardized normal random variables with* $\mathrm{Corr}(Y_{ij}, Y_{i'j'}) = \Lambda_{ij}^{i'j'}$ *such that*

$$\rho := \max_{(i,j)\neq(i',j')} |\Lambda_{ij}^{i'j'}| < 1$$

Then, for $x \geq 0$ and integers $1 \leq d_1 < d_2 < \cdots < d_g \leq n$ and $1 \leq e_1 < e_2 < \cdots < e_h \leq n$ with g, $h \leq n$, we have

$$\begin{aligned} &P\Big\{\max_{1\leq i\leq g}\max_{1\leq j\leq h} Y_{d_i e_j} \leq x\Big\} \\ &\leq \{1-\Phi(x)\}^{gh} + c\sum_{(i,j)\neq(i',j')} |\lambda_{ij}^{i'j'}| \exp\Big(-\frac{x^2}{1+|\lambda_{ij}^{i'j'}|}\Big), \end{aligned} \tag{2.7}$$

where $\lambda_{ij}^{i'j'} = \Lambda_{d_i e_j}^{d_i' e_j'}$ and $c = c(\rho)$ is a positive constant independent of x, n, g and h, and $\Phi(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} \exp(-y^2/2)dy$.

Let us estimate a sharp upper bound for the second term of the right hand side of (2 7)

LEMMA 2.5. *Let $\{Y_{ij}\}, \rho, g, h$ and $\lambda_{ij}^{i'j'}$ be as in Lemma 2.4 Further assume that $|\lambda_{ij}^{i'j'}| < (|i-i'||j-j'|)^{-\mu}$, $i \neq i'$, $j \neq j'$, and set $x = \sqrt{(2-\eta)\log(gh)}$, where μ and η are positive constants such that $0 < \eta < (1-\rho)\mu/\ (1+\mu+\rho)$. Then we have*

$$\sum := \sum_{(i,j)\neq(i',j')} |\lambda_{ij}^{i'j'}| \exp\Big(-\frac{x^2}{1+|\lambda_{ij}^{i'j'}|}\Big) \leq c(gh)^{-\delta},$$

where $\delta = \{\mu(1-\rho) - \eta(1+\rho+\mu)\}/\{(1+\mu)(1+\rho)\} > 0$ and c is a positive constant independent of g, h and n.

Proof. Let $0 < a = (1+\eta\rho-\rho)/\{(1+\mu)(1+\rho)\} < 1$. We split the sum $\sum$ into four parts as follows

$$\begin{aligned}\sum = &\sum_{\substack{1\le i,i'\le g\\ 0<|i-i'|\le[g^a]}}\ \sum_{\substack{1\le j,j'\le h\\ 0<|j-j'|\le[h^a]}} |\lambda_{ij}^{i'j'}|\exp\Big(-\frac{x^2}{1+|\lambda_{ij}^{i'j'}|}\Big)\\ &+\sum_{\substack{1\le i,i'\le g\\ |i-i'|>[g^a]}}\ \sum_{\substack{1\le j,j'\le h\\ |j-j'|>[h^a]}} |\lambda_{ij}^{i'j'}|\exp\Big(-\frac{x^2}{1+|\lambda_{ij}^{i'j'}|}\Big)\\ &+\sum_{\substack{1\le i,i'\le g\\ 0<|i-i'|\le[g^a]}}\ \sum_{\substack{1\le j,j'\le h\\ |j-j'|>[h^a]}} |\lambda_{ij}^{i'j'}|\exp\Big(-\frac{x^2}{1+|\lambda_{ij}^{i'j'}|}\Big)\,.\\ &+\sum_{\substack{1\le i,i'\le g\\ |i-i'|>[g^a]}}\ \sum_{\substack{1\le j,j'\le h\\ |j-j'|\le[h^a]}} |\lambda_{ij}^{i'j'}|\exp\Big(-\frac{x^2}{1+|\lambda_{ij}^{i'j'}|}\Big)\\ =&\sum\nolimits^{(1)}+\sum\nolimits^{(2)}+\sum\nolimits^{(3)}+\sum\nolimits^{(4)},\qquad \text{say.}\end{aligned}$$

Now let us estimate each upper bound of above four sums:

$$\begin{aligned}\sum\nolimits^{(1)} &\le c(gh)^{1+a}\exp\Big(-\frac{2-\eta}{1+\rho}\log(gh)\Big) = c(gh)^{1+a-\{(2-\eta)/(1+\rho)\}}\\ &= c(gh)^{\{\eta(1+\rho+\mu)-\mu(1-\rho)\}/\{(1+\mu)(1+\rho)\}} = c(gh)^{-\delta}\end{aligned}$$

and

$$\begin{aligned}\sum\nolimits^{(2)} &\le c(gh)^2(gh)^{-a\mu}\exp\{-(1-(gh)^{-a\mu})(2-\eta)\log(gh)\}\\ &\le c(gh)^{2-a\mu}\exp\{-(2-\eta)\log(gh)\}\\ &\le c(gh)^{-a\mu+\eta} = c(gh)^{-\delta}.\end{aligned}$$

Also we get

$$\begin{aligned}\sum\nolimits^{(3)} &\le cg^{1+a}\exp\Big(-\frac{2-\eta}{1+\rho}\log g\Big)h^{2-a\mu}\exp\{-(1-h^{-a\mu})(2-\eta)\log h\}\\ &\le cg^{1+a-\{(2-\eta)/(1+\rho)\}}h^{-a\mu+\eta} = c(gh)^{-\delta}.\end{aligned}$$

By the exactly same way, we have

$$\sum\nolimits^{(4)} \le cg^{-a\mu+\eta}h^{1+a-\{(2-\eta)/(1+\rho)\}} = c(gh)^{-\delta}. \qquad \square$$

LEMMA 2.6 *For $k \in \mathbb{N}$, set $T_k = \exp(k^\beta)$ for $\frac{2+\varepsilon}{2+2\varepsilon} < \beta < 1, \varepsilon > 0$, and let T be in $T_k \le T \le T_{k+1}$. Then we have*
(2.8)

$$\liminf_{T\to\infty} \sup_{0\le t\le T-a_T} \sup_{0\le v\le T} \frac{|X(t+a_T,v)-X(t,v)|}{2(a_T)^\alpha\sqrt{\log(T/a_T)+\frac{1}{2}\log\log T}}$$

$$\ge \liminf_{k\to\infty} \sup_{0\le t\le T_k-a_{T_k}} \sup_{0\le v\le T_k} \frac{|X(t+a_{T_k},v)-X(t,v)|}{2(a_{T_k})^\alpha\sqrt{\log(T_k/a_{T_k})+\frac{1}{2}\log\log T_k}} \quad \text{a.s.}$$

Proof. By condition (ii), we have $T - a_T \ge T_k - a_{T_k}$. Put $\mathbb{B}_k = \{(t, a_T, v) : 0 \le t \le T_k - a_{T_k}, a_{T_k} \le a_T \le a_{T_{k+1}}, 0 \le v \le T_k\}$. Then

$$\sup_{0\le t\le T-a_T} \sup_{0\le v\le T} \frac{|X(t+a_T,v)-X(t,v)|}{2(a_T)^\alpha\sqrt{\log(T/a_T)+\frac{1}{2}\log\log T}}$$

$$\ge \left\{ \sup_{0\le t\le T_k-a_{T_k}} \sup_{0\le v\le T_k} \frac{|X(t+a_{T_k},v)-X(t,v)|}{2(a_{T_k})^\alpha\sqrt{\log(T_k/a_{T_k})+\frac{1}{2}\log\log T_k}} \right.$$

$$\left. - \sup_{(t,a_T,v)\in\mathbb{B}_k} \frac{|X(t+a_{T_k},v)-X(t+a_T,v)|}{2(a_{T_k})^\alpha\sqrt{\log(T_k/a_{T_k})+\frac{1}{2}\log\log T_k}} \right\} \frac{(a_{T_k})^\alpha}{(a_{T_{k+1}})^\alpha} \frac{\gamma_{T_k}}{\gamma_{T_{k+1}}}$$

By the same way as in (2.2) and (2.3), we have

$$\lim_{k\to\infty} \frac{(a_{T_k})^\alpha}{(a_{T_{k+1}})^\alpha} \frac{\gamma_{T_k}}{\gamma_{T_{k+1}}} = 1.$$

The proof is completed if we show that

$$\limsup_{k\to\infty} \sup_{(t,a_T,v)\in\mathbb{B}_k} \frac{|X(t+a_{T_k},v)-X(t+a_T,v)|}{(a_{T_k})^\alpha \gamma_{T_k}} = 0 \quad \text{a.s.}$$

In order to apply Lemma 2.1, we set

$$G(t,a_T,v) = \frac{X(t+a_{T_k},v)-X(t+a_T,v)}{(a_{T_k})^\alpha}, \quad (t,a_T,v)\in\mathbb{B}_k,$$

and

$$\varphi(z) = \frac{2(\sqrt{2}z)^\alpha}{(a_{T_k})^\alpha}, \qquad z > 0.$$

Clearly, $E\{G(t, a_T, v)\} = 0$ for all $(t, a_T, v) \in \mathbb{B}_k$. By the same technique as in (2.2), we have, for any $\varepsilon'' > 0$,

$$\Gamma^2 := \sup_{(t,a_T,v)\in\mathbb{B}_k} E\{G(t, a_T, v)\}^2 = \frac{(a_{T_{k+1}} - a_{T_k})^{2\alpha}}{(a_{T_k})^{2\alpha}} \leq (\varepsilon'')^2,$$

provided k is big enough. And, using the elementary relation $(a \pm b)^2 \leq 2(a^2 + b^2)$, we get, for all $(t', a_T{}', v'), (t'', a_T{}'', v'')$ in $\mathbb{B}_k$,

$$\begin{aligned}
&E\{G(t', a_T{}', v') - G(t'', a_T{}'', v'')\}^2 \\
&\leq \frac{1}{(a_{T_k})^{2\alpha}} E\{(X(t' + a_{T_k}, v') - X(t'' + a_{T_k}, v'')) \\
&\qquad - (X(t' + a_T{}', v') - X(t'' + a_T{}'', v''))\}^2 \\
&\leq \frac{2}{(a_{T_k})^{2\alpha}} \Big(\big(\sqrt{(t' - t'')^2 + (v' - v'')^2}\,\big)^{2\alpha} \\
&\qquad + \big(\sqrt{((t' + a_T{}') - (t'' + a_T{}''))^2 + (v' - v'')^2}\,\big)^{2\alpha} \Big) \\
&\leq \frac{4}{(a_{T_k})^{2\alpha}} \big(\sqrt{2}\sqrt{(t' - t'')^2 + (a_T{}' - a_T{}'')^2 + (v' - v'')^2}\,\big)^{2\alpha}.
\end{aligned}$$

Setting $\mathbf{u} = (t', a_T{}', v'), \mathbf{v} = (t'', a_T{}'', v'')$ in $\mathbb{B}_k$, it follows that

$$E\{G(\mathbf{u}) - (\mathbf{v})\}^2 \leq \varphi^2(\|\mathbf{u} - \mathbf{v}\|).$$

For any $\varepsilon' > 0$, there exists a small constant $c > 0$ such that

$$(2\sqrt{2} + 2)A \int_0^\infty \varphi(\sqrt{3}\, c\, a_{T_k}\, 2^{-y^2})\, dy < \varepsilon',$$

where A is a constant such that $A > 2\sqrt{3 \log 2}$. For any given small positive ε and ε', let

$$x = 2\varepsilon \sqrt{\log(T_k/a_{T_k}) + \frac{1}{2} \log\log T_k} \Big/ (\Gamma + \varepsilon').$$

Choosing $\varepsilon' = \varepsilon'' = 2\varepsilon^2$, we have

$$x \geq \sqrt{\log(T_k/a_{T_k}) + \frac{1}{2} \log\log T_k} \Big/ 2\varepsilon.$$

Now applying Lemma 2.1, it follows that

$$
\begin{aligned}
&P\Big\{\sup_{(t,a_T,v)\in\mathbb{B}_k}\frac{|X(t+a_{T_k},v)-X(t+a_T,v)|}{2(a_{T_k})^\alpha\sqrt{\log(T_k/a_{T_k})+\frac{1}{2}\log\log T_k}}>\varepsilon\Big\}\\
&\le P\Big\{\sup_{(t,a_T,v)\in\mathbb{B}_k}\frac{|X(t+a_{T_k},v)-X(t+a_T,v)|}{(a_{T_k})^\alpha}\\
&\qquad > x\Big(\Gamma+(2\sqrt{2}+2)A\int_0^\infty\varphi(\sqrt{3}\,c\,a_{T_k}\,2^{-y^2})\,dy\Big)\Big\}\\
&\le C_\varepsilon\Big(\frac{T_k-a_{T_k}}{a_{T_k}}\vee 1\Big)\Big(\frac{a_{T_{k+1}}-a_{T_k}}{a_{T_k}}\vee 1\Big)\Big(\frac{T_k}{a_{T_k}}\vee 1\Big)\\
&\qquad\times\exp\Big\{-\frac{1}{2}\Big(\frac{\log(T_k/a_{T_k})+\frac{1}{2}\log\log T_k}{4\varepsilon^2}\Big)\Big\}\\
&= C_\varepsilon\Big(\frac{T_k-a_{T_k}}{a_{T_k}}\vee 1\Big)\Big(\frac{a_{T_{k+1}}-a_{T_k}}{a_{T_k}}\vee 1\Big)\Big(\frac{T_k}{a_{T_k}}\vee 1\Big)\Big\{\frac{T_k}{a_{T_k}}\sqrt{\log T_k}\Big\}^{-\frac{1}{8\varepsilon^2}}\\
&\le C_\varepsilon\Big(\frac{T_k}{a_{T_k}}\Big)^{2-\frac{1}{8\varepsilon^2}}(\log T_k)^{-\frac{1}{16\varepsilon^2}}\le C_\varepsilon(\log T_k)^{-\frac{1}{16\varepsilon^2}}=C_\varepsilon k^{-\beta\frac{1}{16\varepsilon^2}}
\end{aligned}
$$

for sufficiently large k. Hence we have

$$
\sum_k P\Big\{\sup_{(t,a_T,v)\in\mathbb{B}_k}\frac{|X(t+a_{T_k},v)-X(t+a_T,v)|}{2(a_{T_k})^\alpha\sqrt{\log(T_k/a_{T_k})+\frac{1}{2}\log\log T_k}}>\varepsilon\Big\}<\infty
$$

and using the Borel-Cantelle lemma,

$$
\limsup_{k\to\infty}\sup_{(t,a_T,v)\in\mathbb{B}_k}\frac{|X(t+a_{T_k},v)-X(t+a_T,v)|}{2(a_{T_k})^\alpha\sqrt{\log(T_k/a_{T_k})+\frac{1}{2}\log\log T_k}}=0\quad\text{a.s.}
$$

This completes the proof of Lemma 2.6. □

We are now ready to prove Theorem 1.2.

Proof of Theorem 1.2. The condition (iii) of Theorem 1.2 implies that one can find $B>0$ and integer $N>0$ large enough such that

$$
\Big(\frac{T-a_T}{a_T}\Big)\Big(\frac{T}{a_T}\Big)>(\log T)^B>N^2 \tag{2.9}
$$

for all large $T > 0$. For such T, we define positive integers g_T and h_T by

$$g_T = \left[\frac{T - a_T}{Na_T}\right] \quad \text{and} \quad h_T = \left[\frac{T}{Na_T}\right],$$

where $[\cdot]$ denotes the integer part. By (2 9), it is clear that there is a constant $c > 0$ such that

$$g_T h_T > c(\log T)^B \tag{2.10}$$

for T large enough. For $i = 0, 1, \cdots, g_T$ and $j = 0, 1, \cdots, h_T$, we also define incremental random variables such that

$$X_{ij} = X(Na_T i + a_T, Na_T j) - X(Na_T i, Na_T j).$$

Then $X_{ij}/(a_T)^\alpha$ are standard normal random variables. It follows from (iii) that, for any $0 < \varepsilon' < \varepsilon < 1$ and large T,

(2.11)

$$\begin{aligned} & P\Big\{\sup_{0\le t\le T-a_T}\sup_{0\le v\le T}\frac{|X(t+a_T,v)-X(t,v)|}{2(a_T)^\alpha\sqrt{\log(T/a_T)+\frac{1}{2}\log\log T}} < \sqrt{1-\varepsilon}\Big\} \\ & \le P\Big\{\sup_{0\le t\le T-a_T}\sup_{0\le v\le T}\frac{X(t+a_T,v)-X(t,v)}{(a_T)^\alpha} < \sqrt{(2-2\varepsilon')\log(g_T h_T)}\Big\} \\ & \le P\Big\{\max_{0\le i\le g_T}\max_{0\le j\le h_T}\frac{X_{ij}}{(a_T)^\alpha} < \sqrt{(2-2\varepsilon')\log(g_T h_T)}\Big\}. \end{aligned}$$

Define the correlation function of X_{ij} and $X_{i'j'}$ as follows:

$$\lambda(i, i', j, j') = \mathrm{Corr}(X_{ij}, X_{i'j'}), \quad i \ne i', \quad j \ne j',$$

and set $l = |i - i'| \ge 1$, $m = |j - j'| \ge 1$. By the relation $ab =$

$\frac{1}{2}(a^2+b^2-(a-b)^2)$, it follows that

$$
\begin{aligned}
&|\mathrm{Cov}(X_{ij}, X_{i'j'})| = |E\{X_{ij}X_{i'j'}\}| \\
&= |E\{X(Na_T i + a_T, Na_T j)X(Na_T i' + a_T, Na_T j') \\
&\quad - X(Na_T i + a_T, Na_T j)X(Na_T i', Na_T j') \\
&\quad - X(Na_T i, Na_T j)X(Na_T i' + a_T, Na_T j') \\
&\quad + X(Na_T i, Na_T j)X(Na_T i', Na_T j')\}| \\
&= \frac{1}{2}|((Na_T l + a_T)^2 + (Na_T m)^2)^\alpha - ((Na_T l)^2 + (Na_T m)^2)^\alpha \\
&\quad - [((Na_T l)^2 + (Na_T m)^2)^\alpha - ((Na_T l - a_T)^2 + (Na_T m)^2)^\alpha]| \\
&\le \frac{1}{2}\left|\int_{\sqrt{(Nl)^2+(Nm)^2}\,a_T}^{\sqrt{(Nl+1)^2+(Nm)^2}\,a_T} d(x^{2\alpha}) - \int_{\sqrt{(Nl-1)^2+(Nm)^2}\,a_T}^{\sqrt{(Nl)^2+(Nm)^2}\,a_T} d(x^{2\alpha})\right|.
\end{aligned}
$$

Using Lemma 2 3 with $a = a_T$ and $q = Nma_T$, we have

$$
|\mathrm{Cov}(X_{ij}, X_{i'j'})| \le c\frac{((Nm)^2 + (Nl+1)^2)^\alpha}{(Nm)^2 + (Nl-1)^2}(a_T)^{2\alpha},
$$

where $c > 0$ is a constant. Hence

$$
\begin{aligned}
|\lambda(i, i', j, j')| &\le c\frac{((Nm)^2 + (Nl+1)^2)^\alpha}{(Nm)^2 + (Nl-1)^2} \\
&\le (l^2 + m^2)^{\alpha-1} \le (2lm)^{\alpha-1} < (lm)^{-\mu},
\end{aligned}
$$

where N is sufficiently large and $\mu = 1 - \alpha > 0$. To estimate an upper bound of the right hand side of (2 11), let us apply Lemmas 2.4 and 2.5 for

$$
\begin{aligned}
&Y_{d_i e_j} = \frac{X_{ij}}{(a_T)^\alpha}, \quad i = 0, 1, \cdots, g_T \;\text{ and }\; j = 0, 1, \cdots, h_T, \\
&|\lambda_{ij}^{i'j'}| = |\lambda(i, i', j, j')| < (lm)^{-\mu}, \quad l = |i - i'| \ge 1, \quad m = |j - j'| \ge 1, \\
&g = g_T, \quad h = h_T, \\
&x = \sqrt{(2-\eta)\log(g_T h_T)}, \quad \eta = 2\varepsilon' < (1-\rho)\mu/(1+\mu+\rho)
\end{aligned}
$$

Then the right hand side of (2.11) is less than or equal to

$$
\{1 - \Phi(x)\}^{g_T h_T} + c(g_T h_T)^{-\delta}.
$$

Thus we have

(2.12)
$$P\Big\{\sup_{0\le t\le T-a_T}\sup_{0\le v\le T}\frac{|X(t+a_T,v)-X(t,v)|}{2(a_T)^\alpha\sqrt{\log(T/a_T)+\frac{1}{2}\log\log T}}<\sqrt{1-\varepsilon}\Big\}$$
$$\le \exp\big(-C(g_T h_T)^\varepsilon\big)+c(g_T h_T)^{-\delta}$$
$$\le c(g_T h_T)^{-\delta},$$

where C and c are positive constants changing in lines. For $k\in\mathbb{N}$, set $T_k=\exp(k^\beta)$, $(2+\varepsilon)/(2+2\varepsilon)<\beta<1$. Note that the B in (2.9) can be taken so large that $B>1/(\beta\delta)$ for given β and δ. Then (2.10) and (2.12) yield

$$P\Big\{\sup_{0\le t\le T_k-a_{T_k}}\sup_{0\le v\le T_k}\frac{|X(t+a_{T_k},v)-X(t,v)|}{2(a_{T_k})^\alpha\sqrt{\log(T_k/a_{T_k})+\frac{1}{2}\log\log T_k}}<\sqrt{1-\varepsilon}\Big\}$$
$$\le c\,k^{-\beta\delta B}$$

and the series

$$\sum_k P\Big\{\sup_{0\le t\le T_k-a_{T_k}}\sup_{0\le v\le T_k}\frac{|X(t+a_{T_k},v)-X(t,v)|}{2(a_{T_k})^\alpha\sqrt{\log(T_k/a_{T_k})+\frac{1}{2}\log\log T_k}}<\sqrt{1-\varepsilon}\Big\}$$

is convergent. So the Borel-Cantelli lemma implies that

$$\liminf_{k\to\infty}\sup_{0\le t\le T_k-a_{T_k}}\sup_{0\le v\le T_k}\frac{|X(t+a_{T_k},v)-X(t,v)|}{2(a_{T_k})^\alpha\sqrt{\log(T_k/a_{T_k})+\frac{1}{2}\log\log T_k}}\ge 1 \qquad \text{a.s.}$$

Thus Theorem 1.2 follows immediately from (2.8) of Lemma 2.6. □

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