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DEGREE OF THE GAUSS MAP ON AN ODD DIMENSIONAL MANIFOLD

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ABSTRACT For a codimension 1 submanifold in a Euclidean $2n$-space, the degree of the gauss map mod 2 is the semi-characteristic of the manifold in Z_2 coefficient.

1. Introduction

Let W be a compact smooth $(2n+1)$-manifold with boundary M. Then the Euler number $\chi(W)$ is the same as the half of $\chi(M)$ This can easily be seen by the long exact sequence for the pair (W, M) together with the Poincaré duality: $H^*(W, M) \cong H_{2n-*+1}(W)$. This is one of the essential reasons why the degree of the Gauss map for a codimension 1 embedding, $M \to R^{2n+1}$, is $\frac{1}{2}\chi(M)$ regardless of the specific embedding(see 5.2 below).

However, when W is of even dimension, the corresponding relation does not hold in general between the Euler number of W and that of its boundary Restricting to the parallelizable case, we will prove *1.1* below.

Let $\chi_{\frac{1}{2}}(M)$ denote the semi-characteristic in Z_2-coefficient, which is defined as

$$\sum_{i=0}^{n-1} rank\ H_i(M; Z_2), \quad mod\ 2$$

for any closed $(2n-1)$-manifold M.

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THEOREM 1.1. *If W is a compact parallelizable manifold of even dimension with boundary M, then we have:*

$$\chi(W) \equiv \chi_{\frac{1}{2}}(M) \quad mod\ 2.$$

This together with *5.2* below proves the following:

THEOREM 1.2. *Let M be a closed smooth $(2n-1)$-manifold with an embedding into R^{2n}. Then the degree of the associated Gauss map is the same as $\chi_{\frac{1}{2}}(M)$ mod 2.*

That is, for an odd dimensional manifold which admits a codimension 1 embedding into a Euclidean space, the parity of the degree of the Gauss map does not depend on the choice of a codimension 1 embedding.

The proof of *1.1* would have been much simpler if we restricted ourselves to the case when the dimension of M is other than $1, 3, 7$ and if we exploited some statements by Dupont (*3.2*, *3.4*, [5]). In this paper, we do not use his results even if our use of Wu-cospectrum of spheres is inspired by his work. Also we are helped by the idea of 'symmetric lifting' due to Sutherland([8]), which however appears in a quite different form(see §3, 4 below). As a whole, the technique of the paper is motivated by the works of Dupont([5,6]), Sutherland([8]) in the forms slightly refined by the author ([3,4]).

2. Browder's quadratic function

What follows in this section is a special case of W. Browder's work([1]). We will concentrate on the case when the Wu-cospectrum is that of spheres. For more details one must refer to Browder's work itself.

Let S^l denote the standard l-sphere, $\Sigma^k X$, the k-fold reduced suspension for any pointed space X. For any space Y, let Y_+ denote Y together with an additional point $+$ which acts as the base point. Let $T(\xi)$ denote the Thom space for any vector bundle ξ. Write K_n to denote an n-th Eilenberg-Mac lane space $K(Z_2, n)$.

Assume N is a stably parallelizable closed $2n$-manifold, which is not necessarily connected. Let $k > 2n+1$ and $\iota : N \to S^{2n+k}$ be a smooth embedding. Let ν_ι denote the normal bundle, which is understood as the pull-back of a subbundle of the tangent vector bundle of S^{2n+k}. Also choose a trivialization, $\theta : \nu_\iota \to \epsilon^k$. Let $c : S^{2n+k} \to T(\nu_\iota)$ denote the collapse map given by choosing a normal neighborhood of N.

Then there is a map $f : S^{2n+k} \to \Sigma^k N_+$ which is the composite:

$$S^{2n+k} \xrightarrow{c} T(\nu_\iota) \overset{T(\theta)}{\cong} T(\epsilon^k) = \Sigma^k N_+ \ .$$

We may regard f as a part of a Wu-orientation of N understanding S^{2n+k} as a part of a Wu-(n+1)-cospectrum in the language of Browder.

Then f defines a quadratic function:

$$\psi : H^n(N; Z_2) \to Z_2 \ .$$

In fact, ψ is defined as follows: Let $v \in H^n(N; Z_2)$. Then v determines a homotopy class of maps, $N \to K_n$, which we denote by v again. Write δ for the composite:

$$S^{2n+k} \xrightarrow{f} \Sigma^k N_+ \xrightarrow{\Sigma^k v} \Sigma^k K_n \ .$$

Then $\psi(v)$ is defined as $Sq^{n+1}_\delta(\Sigma^k a) \in H^{2n+k}(S^{2n+k}; Z_2) \cong Z_2$, where $a \in H^n(K_n; Z_2)$ is the generator and Sq^{n+1}_δ means the functional Steenrod square.

Note that ψ is *quadratic* in the sense:

$$\psi(v + w) = \psi(v) + \psi(w) + v \cdot w$$

for any $v, w \in H^n(N; Z_2)$, where $v \cdot w$ means the intersection number.

As E. H. Brown has shown(*1.2*, [2]), the group of homotopy classes of maps, $[S^{2n+k}, \Sigma^k K_n]$, is isomorphic to Z_2. In addition, there are examples in which $\psi(v) \neq 0$ (for example, see §3 below). Therefore, we have:

(2.1) *δ above is null homotopic if and only if $\psi(v) = 0$.*

Let O_k denote the group of orthogonal matrices of rank k. We observe the following two obvious facts, which we prepare for future uses (see (4.1) below and Proof of 1.1 in §5):

(2.2) *Let $\theta' : \nu \to \epsilon^k$ be another trivialization and $\alpha : \epsilon^k \to \epsilon^k$ be the isomorphism such that $\theta' = \alpha\theta$. Let ψ' denote the quadratic function* *obtained by replacing* θ *with* θ'. *Let* $\bar{\alpha} : N \to O_k$ *denote the map defined by* α. *Then, if* $\bar{\alpha}$ *is homotopic to a constant map when restricted to each component of* N, *then we have that* $\psi' = \psi$.

(2.3) *Let* N_1 *be a component of* N *and let* $\iota_1 : N_1 \to S^{2n+k}$, ν_1 *and* $\theta_1 : \nu_1 \to \epsilon^k_{N_1}$ *be the restrictions. If* ψ_1 *denotes the associated quadratic function,* $H^n(N_1; Z_2) \to Z_2$, *then we have that* $\psi_1 = \psi \mid_{H^n(N_1,Z_2)}$.

3. Evaluation at the diagonal class

If M is a closed n-manifold with fundamental class $[M]$, then the product $M \times M$ is a closed $2n$-manifold with fundamental class $[M] \times [M]$. The diagonal (cohomology) class, $u \in H^n(M \times M; Z_2)$, is defined as the unique cohomology class satisfying:

$$\Delta_*[M] = ([M] \times [M]) \cap u$$

where $\Delta : M \to M \times M$ means the diagonal map.

Assume n is odd. Let $t : M \times M \to M \times M$ denote the map transposing the factors. Then we have: $u = a + t^*a$ for some $a \in H^n(M \times M; Z_2)$ such that $a \cdot t^*a = \chi_{\frac{1}{2}}(M)$ (cf. the proof of 6.1, [3]).

Furthermore, assume M is a stably parallelizable closed submanifold of S^{n+k}, $k > n+1$. Let ν_M denote the normal vector bundle of M and $\theta_M : \nu_M \to \epsilon^k_M$ denote a trivialization.

Now we may regard $M \times M$ as a submanifold of $S^{2n+2k+1}$ which is, up to a diffeomorphism, identified with $S^{n+k} \times D^{n+k+1} \cup D^{n+k+1} \times S^{n+k} = \partial(D^{n+k+1} \times D^{n+k+1})$. Here, $D^{n+k+1} \times D^{n+k+1}$ is understood as a smooth manifold up to rounding off the corner.

Since $M \times M \subset S^{n+k} \times S^{n+k} \subset \partial(D^{n+k+1} \times D^{n+k+1})$, we have $\nu_{M\times M} = \nu_M \times \nu_M + \epsilon^1$. Here ϵ^1 is understood as the restriction to $M \times M$ of the normal vector bundle of $S^{n+k} \times S^{n+k}$ in $\partial(D^{n+k+1} \times D^{n+k+1})$ up to a fixed isomorphism. Then there is the trivialization:

$$\theta_M \times \theta_M + 1 : \nu_M \times \nu_M + \epsilon^1 \to \epsilon^k_M \times \epsilon^k_M + \epsilon^1$$

which we call the product trivialization and denote by θ.

Now let $f : S^{2n+2k+1} \to \Sigma^{2k+1} M \times M_+$ be the Wu-$(n+1)$-orientation determined by θ and the natural collapse map, $S^{2n+2k+1} \to T(\nu_{M\times M})$. In the following diagram, let t denote the maps transposing the factors.

$$\begin{array}{ccc} S^{2n+2k+1} & \xrightarrow{f} & \Sigma^{2k+1}M \times M_+ \\ t\downarrow & & \bar{t}\Sigma^{2k+1}t\downarrow \\ S^{2n+2k+1} & \xrightarrow{f} & \Sigma^{2k+1}M \times M_+ \end{array}$$

This diagram commutes, where $\bar{t}$ denotes the map, for any space X with a base point,

$$\Sigma^{2k+1}X = ((S^k \wedge S^k) \wedge S^1) \wedge X \xrightarrow{(t\wedge r)\wedge 1} ((S^k \wedge S^k) \wedge S^1) \wedge X = \Sigma^{2k+1}X$$

in which r means a reflection.

Considering (2.1), if we use θ above to construct the Wu-$(n+1)$-orientation and, subsequently, the quadratic function ψ on $H^n(M \times M; Z_2)$, we have $\psi(v) = \psi(t^*v)$ for any $v \in H^n(M \times M; Z_2)$. Therefore, for the diagonal class u, we have:

$$\psi(u) = \chi_{\frac{1}{2}}(M) \tag{3.1}$$

which can be seen easily by the equalities:

$$\psi(u) = \psi(a + t^*a) = \psi(a) + \psi(t^*a) + a \cdot t^*a = a \cdot t^*a = \chi_{\frac{1}{2}}(M)$$

4. Dupont-Sutherland invariant

Let M be a stably parallelizable closed manifold of odd dimension n. Let ξ be a vector bundle over M of rank n which is stably trivial and $\theta : \epsilon^{n+k} \to \xi + \epsilon^k$ be a bundle isomorphism, $k > n+1$. Note that there is a degree one map, say, $f : S^{2n+k} \to T(\epsilon^{n+k})$. Let U denote the Thom class of ξ. Then, we consider the following composite which we call δ:

$$S^{2n+k} \xrightarrow{f} T(\epsilon^{n+k}) \xrightarrow{T(\theta)} T(\xi + \epsilon^k) = \Sigma^k T(\xi) \xrightarrow{\Sigma^k U} \Sigma^k K_n \ .$$

Write $b(\xi) = Sq_\delta^{n+1}a \in Z_2$, which is just the Dupont-Sutherland invariant for ξ(cf. [5,8]). In general, $b(\xi)$ depends on the choice of f and θ unless there are two isomorphism classes of stably trivial vector bundles of rank n over M (cf.[5,6]).

In a similar spirit to (2.2), we have:

(4.1) *Let $\theta' : \epsilon^{n+k} \to \xi' + \epsilon^k$ be another isomorphism and $b(\xi')$ be obtained by replacing θ with θ' in the above. If there is an isomorphism, $\alpha : \xi \to \xi'$, such that ${\theta'}^{-1}(\alpha + 1)\theta : \epsilon^{n+k} \to \epsilon^{n+k}$ induces a map $M \to O_{n+k}$ which is homotopic to a constant map when restricted to each component of M. Then we have: $b(\xi') = b(\xi)$.*

Let τ_N denote the tangent vector bundle for any smooth manifold N.

Assume $M = \Delta M \subset M \times M \subset S^{2n+k}$. In the previous section, we have observed that there is a trivialization $\theta : \nu^k_{M\times M} \to \epsilon^k_{M\times M}$ which, together with the natural collapse map $c : S^{2n+k} \to T(\nu^k_{M\times M})$, result in a quadratic function, $\psi : H^n(M \times M; Z_2) \to Z_2$, satisfying $\psi = \psi t^*$.

We may identify τ_M with the normal bundle of ΔM in $M \times M$. Thus, we have a vector bundle isomorphism α which is the composite:

$$\nu_M = \tau_M + \nu' \xrightarrow{1+\theta'} \tau_M + \epsilon^k$$

where ν_M denotes the normal bundle of $M = \Delta M$ in S^{2n+k} and ν' is the restriction of $\nu_{M\times M}$ at $M = \Delta M$ and θ' is the restriction of θ. Together with the natural collapse map $c' : S^{2n+k} \to T(\nu^{n+k}_M)$, by applying (3.1) we have that:

$$b(\tau_M) = \psi(u) = \chi_{\frac{1}{2}}(M) \tag{4.2}$$

Here u means the diagonal class. To be more precise, (4.2) follows from the following two facts.

First of all, the composite,

$$S^{2n+k} \xrightarrow{c'} T(\nu_M) = T(\tau_M + \nu') \xrightarrow{T(1+\theta')} T(\tau_M + \epsilon^k_M) = \Sigma^k T(\tau_M)\ ,$$

is homotopic to the composite,

$$S^{2n+k} \xrightarrow{c} T(\nu_{M\times M}) \xrightarrow{T(\theta)} T(\epsilon^k_{M\times M}) = \Sigma^k M \times M_+ \xrightarrow{\Sigma^k c''} \Sigma^k T(\tau_M$$

where $c'' : M \times M_+ \to T(\tau_M)$ means the collapse map.

Secondly, the pull-back of the Thom class $U \in H^n(T(\tau_M); Z_2)$ by c'' is the diagonal class $u \in H^n(M \times M; Z_2)$. (cf. [5,6] or [8]).

5. Proofs of main theorems

For any non-vanishing section s of a vector bundle η over a space X, let ϵ_s denote the rank 1 subbundle of η determined by s. If s' is another such section for a vector bundle η' over X, then we will, slightly abusing the notation, let $1 : \epsilon_s \to \epsilon_{s'}$ denote the bundle map which maps $s(x)$ to $s'(x)$ for each $x \in X$.

The following is a key lemma to prove *1.1*.

LEMMA 5.1. *Let W be a parallelizable compact n-manifold with boundary M where n is even. Let μ denote a non-vanishing vector field of $\tau_W \mid_M$ pointing outward. Assume that there is a non-vanishing tangent vector field s of τ_W such that, if ξ denotes a subbundle of τ_W satisfying $\xi + \epsilon_s = \tau_W$, there is an isomorphism $\alpha : \xi \mid_M \to \tau_M$. Furthermore, let the map $\overline{\alpha + 1} : M \to O_n$ be the one determined by the composite,*

$$\tau_W \mid_M = \xi \mid_M + \epsilon_s \xrightarrow{\alpha+1} \tau_M + \epsilon_\mu = \tau_W \mid_M$$

together with a trivialization $\tau_W \mid_M \to \epsilon^n$. Then, if $\overline{\alpha + 1}$ is homotopic to a constant map when restricted to each component of M, we have: $\chi_{\frac{1}{2}}(M) = 0$.

Proof. Consider the following diagram:

$$\begin{array}{ccccccc}
S^{2n+2k-1} & \longrightarrow & \Sigma^{2k}T(\tau_W \mid_M) & = & \Sigma^{2k}T(\tau_M + \epsilon_\mu) & \longrightarrow & \Sigma^{2k+1}K_{n-1} \\
\| & & \downarrow{\scriptstyle x} & & \downarrow{\scriptstyle T(\alpha+1)} & & \downarrow \\
S^{2n+2k-1} & \longrightarrow & \Sigma^{2k}T(\tau_W \mid_M) & = & \Sigma^{2k}T((\xi + \epsilon_s) \mid_M) & \longrightarrow & \Sigma^{2k+1}K_{n-1} \\
\downarrow & & \downarrow & & \downarrow & & \| \\
D^{2n+2k} & \longrightarrow & \Sigma^{2k}T(\tau_W) & = & \Sigma^{2k}T(\xi + \epsilon_s) & \longrightarrow & \Sigma^{2k+1}K_{n-1}
\end{array}$$

where x is either the identity or a reflection of the suspension part when restricted to each component of M (see (4.1)) and the vertical arrows without labels are inclusions. Note that we have $\Sigma^{2k}T(\eta+\epsilon_1) = \Sigma^{2k+1}T(\eta)$ for any vector bundle η. Furthermore, the map $D^{2n+2k} \to$

$\Sigma^{2k}T(\tau_W)$ and its restriction $S^{2n+2k-1} \to \Sigma^{2k}T(\tau_W\mid_M)$ are given as follows: Consider an embedding $\iota : W \to D^{n+k}$ such that ι restricted to some collar neighborhoods, $M \times [0,1] \subset W$ and $S^{n+k-1} \times [0,1] \subset D^{n+k}$, is a product embedding, say, $\iota_1 \times 1$. Then, $\iota \times \iota : W \times W \to D^{n+k} \times D^{n+k} \cong D^{2n+2k}$ is a smooth embedding up to rounding off the corners using the collar neighborhoods. Note that τ_W is isomorphic to a normal bundle of the diagonal embedding $W \to W \times W$. We choose a trivialization $\nu_\iota \to \epsilon^k$ and use the product trivialization(see §3):

$$\nu_{\iota\times\iota} = \nu_\iota \times \nu_\iota \to \epsilon^k \times \epsilon^k = \epsilon^{2k}_{W\times W} .$$

Using these data we define $D^{2n+2k} \to \Sigma^{2k}T(\tau_W)$ as the composite:

$$D^{2n+2k} \to T(\nu_{\iota\times\iota}) \to T(\epsilon^{2k}_{W\times W}) \cong \Sigma^{2k}(W \times W_+) \to \Sigma^{2k}T(\tau_W) .$$

Then the map $S^{2n+2k-1} \to \Sigma^{2k}T(\tau_W\mid_M)$ is just the restriction. The arrows into $\Sigma^{2k+1}K_{n-1}$ represent the suspensions of the Thom classes.

Then the rectangles commute except for the upper second one which commutes up to homotopy for a proper choice of x(see (4.1)).

As observed by (4.2), $\chi_{\frac{1}{2}}(M) = b(\tau_M)$ where $b(\tau_M)$ is given by the first row of the above diagram. Furthermore, commutativity of the above diagram means: $b(\tau_M) = b(\xi\mid_M)$ and $b(\xi\mid_M) = 0 \in Z_2$(see (2.1)). This completes the proof.

□

We keep the notations of *5.1* above. Consider W with a framing $F : \tau_W \to R^n$. Then F together with μ defines a Gauss map $g : M \to S^{n-1}$ by $g(x) = F(\mu_x)/ \parallel F(\mu_x) \parallel$. Assume F respects the orientations and M is oriented so that the orientation of M followed by μ_x is the orientation of W at each $x \in M$ and $S^{n-1} \subset D^n$ is oriented similarly when $D^n \subset R^n$ is given the standard orientation. Recall that

$$\deg\ g = \int_M g^*\Omega_{n-1}$$

where Ω_{n-1} is the volume form of S^{n-1} divided by the volume of S^{n-1}.

LEMMA 5.2. $\deg g = \chi(W)$

Proof. Choose a Morse function $h : W \to [0,1]$ for the triad $(W; \emptyset, M)$ and a gradient-like vector field s for h (cf. p.20, [7]) so that
i) $s(f) > 0$ throughout the complement of the set of all critical points of h and
ii) given any critical point p, there is an oriented coordinate system

$$(x,y) = (x_1, \cdots, x_\lambda, y_1, \cdots, y_{n-\lambda})$$

on a neighborhood U of p so that $h(q) = f(p) - \mid x(q) \mid^2 + \mid y(q) \mid^2$ and $s(q)$ has the coordinate $(-x(q), y(q))$ for all $q \in U$.

Furthermore we may assume $s \mid_M = \mu$.

Choose, for each critical point p, a small $\varepsilon > 0$ and write $D_p^n = \{q \in U \mid\mid x(q) \mid^2 + \mid y(q) \mid^2 \leq \varepsilon^2\}$ so that D_p^n is diffeomorphic to the closed ε-ball centered at $O \in R^n$, by the restriction of the coordinate system.

Let W_0 be the manifold obtained by deleting from W the interior of D_p^n for each critical point p.

Then we have a map $\bar{g} : W_0 \to S^{n-1}$ defined by $\bar{g}(q) = F(s(q))/ \parallel F(s(q)) \parallel$, which extends $g : M \to S^{n-1}$.

By Stoke's theorem we have:

$$\int_{\partial W_0} \bar{g}^* \Omega_{n-1} = \int_{W_0} d\bar{g}^* \Omega_{n-1} = \int_{W_0} \bar{g}^*(d\Omega_{n-1}) = 0 .$$

Write $S_p^{n-1} = \partial D_p^n$ for each critical point p of h and orient S_p^{n-1} in the standard way using the orientation of $D_p^n \subset W$. Then we have:

$$\partial W_0 = M \cup (\cup_p - S_p^{n-1})$$

where p runs through all the critical points of h.

Thus we have:

$$\deg g = \int_M \bar{g}^* \Omega_{n-1} = \sum_p \int_{S_p^{n-1}} \bar{g}^* \Omega_{n-1} = \sum_p \deg \bar{g} \mid_{S_p^{n-1}} .$$

Now it is straightforward to see that $\deg \bar{g} \mid_{S_p^{n-1}} = (-1)^{\lambda_p}$ where λ_p is the index of the critical point p of h.

Therefore, we conclude $\deg g = \sum_p (-1)^{\lambda_p} = \chi(W)$, as desired.

□

Proof of 1.1. First of all we show that the compact manifold W_0 and the gradient like vector field s on W_0, which are constructed as in the proof of *5.2* satisfies all the conditions of *5.1*.

The restriction of s at M is just $\mu\,|_M$. Therefore, we may choose ξ so that $\xi\,|_M = \tau_M$.

Let p be a critical point of index λ for the Morse function on W and $(x,y) = (x_1, \cdots, x_\lambda, y_1, \cdots, y_{n-\lambda})$ be the coordinate system for a neighborhood of p. Then (x,y) defines a framing for $\tau_{W_0}\,|_{S_p^{n-1}}$ so that

$$s(q) = (-x_1(q), \cdots, -x_\lambda(q), y_1(q), \cdots, y_{n-\lambda}(q))_q \ ,$$

for any $q \in S_p^{n-1}$. Furthermore, we have $\mu(q) = (-x(q), -y(q))_q$ and the tangent vectors to S_p^{n-1} at q are the vectors $(v,w)_q$ such that $x \cdot v + y \cdot w = 0$. We may choose the subbundle ξ so that ξ_q consists of the vectors $(v, -w)_q$ where $(v,w)_q$ is a tangent vector to S_p^{n-1}. Then we define $\alpha : \tau_{S_p^{n-1}} \to \xi\,|_{S_p^{n-1}}$ by $\alpha((v,w)_q) = (v,-w)_q$ and the same formula maps $\mu(q)$ onto $s(q)$. Thus $\alpha + 1 : \tau_W \to \tau_W$ is the bundle map that maps $(v,w)_q$ to $(v,-w)_q$ for any $(v,w)_q \in (\tau_W)_q$.

Now apply *5.1* to conclude that $\chi_{\frac{1}{2}}(\partial W_0) = 0$. Also we have that $\partial W_0 = M \cup (\cup_p S_p^{n-1})$ and $\chi_{\frac{1}{2}}(S_p^{n-1}) = 1 \in Z_2$ for any critical point p. Therefore, considering (2.3), we conclude:

$$\chi_{\frac{1}{2}}(M) = \sum_p \chi_{\frac{1}{2}}(S_p^{n-1}) = \sum_p 1 \in Z_2 \ .$$

Finally, the proof of *1.1* is complete by observing:

$$\sum_p 1 \equiv \sum_p (-1)^{\lambda_p} = \chi(W) \ \ mod\ 2 \ .$$

□

Proof of 1.2. Consider the closure W of the bounded component of $R^{2n} - M$. Then $\chi(W) \equiv \chi_{\frac{1}{2}}(M)$ mod 2 by *1.1*. The degree of the Gauss map associated to the natural framing of $W \subset R^{2n}$ and the outward unit normal on M is $\chi(W)$ by *5.2*, which completes the proof.

□

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