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NOTES ON THE ANALYTIC FEYNMAN INTEGRAL

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ABSTRACT In this paper, we prove a translation theorem for the analytic Feynman integral for functions in $\mathcal{F}_{A_1,A_2}(B)$ and show how this integral can be modified so as to be translation invariant.

1.Introduction

Let H be a real separable, infinite dimensional Hilbert space and let $M(H)$ denote the space of all C-valued countably additive measures on $\mathcal{B}(H)$, the Borel class of H. The Fresnel class $\mathcal{F}(H)$ is defined as the space of all Fourier transforms of elements of $M(H)$ [2]; that is, $\mathcal{F}(H) = \{f = \hat{\mu} : \mu \in M(H)\}$ where

$$(1.1) \qquad f(h) = \int_H \exp\{i < h, h_1 >\} d\mu(h_1).$$

Also if we define $\|f\| = \|\mu\|$, then $\mathcal{F}(H)$ becomes a Banach algebra isomorphic to $M(H)$. Finally, the Fresnel integral $\mathcal{F}(f)$ is defined by

$$(1.2) \qquad \mathcal{F}(f) = \int_H \exp\{-\frac{i}{2}\|h\|^2\} d\mu(h).$$

When appropriate choices are made for the Hilbert space H and the function f, the Fresnel integral provides one means of giving a rigorous definition of Feynman's path integral from quantum theory [2,8]. The Fresnel integral has defined on Hilbert space [2] , classical Wiener space [3] and abstract Wiener space (H, B, ν) [11,12] settings, and used as an approach to the Feynman integral.

Recently, Kallianpur and Bromley [11] introduced the analytic Feynman integral on abstract Wiener space, and established existence theorems of this integral for functions in the Fresnel class $\mathcal{F}(B)$. And

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Kallianpur, Kannan and Karandikar [12] also introduced the sequential Feynman integral on abstract Wiener and Hilbert spaces, and obtained existence theorems of the analytic and the sequential Feynman integrals for functions in larger classes $\mathcal{G}^q(B)$ and $\mathcal{G}^q(H)$ than Fresnel classes $\mathcal{F}(B)$ and $\mathcal{F}(H)$ considered in [2,11]. Moreover, Chung and Kang [7] also proved the translation theorem of the above integrals for functions in $\mathcal{G}^q(B)$ and $\mathcal{G}^q(H)$.

In this paper, we introduce the analytic Feynman integral on $B \times B$ discussed by Kallianpur and Bromley [11], and prove a translation theorem of this integral for functions in larger classes $\mathcal{F}_{A_1,A_2}(B)$ and $\mathcal{F}_{A_1,A_2}(H)$ than $\mathcal{F}(B)$ and $\mathcal{F}(H)$, which differ from the classes $\mathcal{G}^q(B)$ and $\mathcal{G}^q(H)$. Moreover we show how the integral can be modified so as to be translation invariant.

2. Preliminaries

Let H be a real separable infinite dimensional Hilbert space with inner product $< \cdot,\cdot >$ and norm $|\cdot|$. Let m be the Gauss measure on H defined by

$$m(E) = (2\pi)^{-\frac{n}{2}} \int_F \exp\{-\frac{|x|^2}{2}\}dx$$

where $E = P^{-1}(F)$, F is a Borel set in the image of an n-dimensional projection P in H and dx is Lebesgue measure in PH. Let $\|\cdot\|$ be a measurable norm on H with respect to m on H. Let B denote the completion of H with respect to $\|\cdot\|$ and let i denote the natural injection from H to B. The adjoint operator i^* of i is one-to-one and maps B^* continuously onto a dense subset of H^*. By identifying H with H^* and B^* with $i^*(B^*)$, we have a triple $B^* \subset H^* \equiv H \subset B$ and $< h, x >= (h, x)$ for all $h \in H$ and $x \in B^*$, where $(\cdot,\cdot)$ denotes the natural dual pairing between B and B^*. By a well known result of Gross [9], $m \circ i^{-1}$ has a unique countably additive extension ν to the Borel σ-algebra $\mathcal{B}(B)$ of B. The triple (H, B, ν) is called an abstract Wiener space and the Hilbert space H is called the generator of (H, B, ν). For more detail, see [9,10,11,12,13].

Let $\{e_n\}$ denote a complete orthogonal system on H such that the e_n's are in B^*. For each $h \in H$ and $x \in B^*$, we introduce a stochastic inner product on $H \times B$ defined by

$$(2.1)\quad (h,x)^{\sim} = \begin{cases} \lim\limits_{n\to\infty} \sum_{k=1}^{n} < h, e_k > (e_k, x), & \text{if the limit exists} \\ 0, & \text{otherwise.} \end{cases}$$

It is well known that for every $h \in H$, $(h,x)^{\sim}$ exists for ν-a.e. $x \in B$ and is a Borel measurable function of B having a Gaussian distribution with mean zero and variance $\|h\|^2$. Furthermore, it is easy to show that $(h,x)^{\sim} = (h,x)$ ν-a.e. on B if $h \in B^*$. Note that if both h and x are in H then $(h,x)^{\sim} =< h, x >$.

Given two C-valued functions F and G on $B \times B$, we say that $F = G$ s-a.e. if $F(\alpha x_1, \beta x_2) = G(\alpha x_1, \beta x_2)$ for $\nu \times \nu$-a.e. $(x_1, x_2) \in B \times B$ for all $\alpha > 0$ and $\beta > 0$. For a function F on $B \times B$, we will denote by [F] the equivalence class of functions which equal s-a.e..

Let $M(H)$ denote the class of C-valued countably additive measures defined on $\mathcal{B}(H)$. $M(H)$ is a Banach algebra under the total variation norm and with convolution as multiplication.

DEFINITION 2.1. *Let A_1 and A_2 be two bounded, non-negative self-adjoint operators on H. Let $\mathcal{F}_{A_1,A_2}(B)$ be the space of all s-equivalence classes of functions F on $B \times B$ which, for some $\mu \in M(H)$, have the form*

$$(2.2)\qquad F(x_1, x_2) = \int_H \exp\{i[(A_1^{\frac{1}{2}} h, x_1)^{\sim} + (A_2^{\frac{1}{2}} h, x_2)^{\sim}]\} d\mu(h).$$

The map $\mu \to [F]$ defined by (2.2) sets up an algebra isomorphism between $M(H)$ and $\mathcal{F}_{A_1,A_2}(B)$ if $Ran(A_1 + A_2)$ is dense in H [11,p241] where Ran indicates the range of an operator. In this case, $\mathcal{F}_{A_1,A_2}(B)$ becomes a Banach algebra under the norm $\|F\| = \|\mu\|$.

DEFINITION 2 2. *Let F be a functional on $B \times B$ such that the integral*

$$(2.3)\qquad J_F(\lambda_1, \lambda_2) = \int_{B\times B} F(\lambda_1^{-\frac{1}{2}} x_1, \lambda_2^{-\frac{1}{2}} x_2) d(\nu \times \nu)(x_1, x_2)$$

exists for all $\lambda_1 > 0$ and $\lambda_2 > 0$. If there exists an analytic function $J_F^*(z)$ on $\Omega = \{z = (z_1, z_2) \in C^2 : Re z_k > 0$ for $k = 1, 2\}$ such that

$J_F^*(\lambda_1, \lambda_2) = J_F(\lambda_1, \lambda_2)$ *for all* $\lambda_1 > 0$ *and* $\lambda_2 > 0$, *then we define* $J_F^*(z)$ *to be the analytic Wiener integral of* F *over* $B \times B$ *with parameter* z *and, for* $z \in \Omega$, *we write*

$$I_a^z[F(\cdot,\cdot)] = J_F^*(z). \tag{2.4}$$

Let q_1 *and* q_2 *be non-zero real numbers. If the following limit (2.5) exists, we call it the analytic Feynman integral of* F *over* $B \times B$ *with parameter* $q = (q_1, q_2)$ *and we write*

$$I_a^q[F(\cdot,\cdot)] = \lim_{z \to (-iq_1, -iq_2)} I_a^z[F(\cdot,\cdot)] \tag{2.5}$$

where $z = (z_1, z_2)$ *approaches through* Ω. *In particular, if* $q_1 = q_2 = q$, *we write*

$$\mathcal{I}_a^q[F(\cdot,\cdot)] = I_a^{(q,q)}[F(\cdot,\cdot)]. \tag{2.6}$$

The following theorem plays an important role in this paper. We state it without proof [11].

THEOREM 2.3. *Let* $F \in \mathcal{F}_{A_1,A_2}(B)$ *be given by (2.2). Then the analytic Feynman integral* $I_a^q[F(\cdot,\cdot)]$ *exists for* $q = (q_1, q_2)$ *where* $q_1 \neq 0$ *and* $q_2 \neq 0$, *and*

$$I_a^q[F(\cdot,\cdot)] = \int_H \exp\{-\frac{i}{2}\sum_{k=1}^{2} q_k^{-1} < A_k h, h >\} d\mu(h). \tag{2.7}$$

In particular,

$$I_a^{(1,-1)}[F(\cdot,\cdot)] = \int_H \exp\{-\frac{i}{2} < (A_1 - A_2)h, h >\} d\mu(h). \tag{2.8}$$

REMARK 2.4. Let A be a bounded self-adjoint operator on H. Then we may write $A = A^+ - A^-$ where A^+ and A^- are each bounded, non-negative and self-adjoint. Take $A_1 = A^+$ and $A_2 = A^-$ in the definition above. For any $F \in \mathcal{F}_{A_1,A_2}(B)$, (2.8) becomes

$$I_a^{(1,-1)}[F(\cdot,\cdot)] = \int_H \exp\{-\frac{i}{2} < Ah, h >\} d\mu(h)$$

Also, in this case, if A^+ is the identity and $A^- = 0$, then $\mathcal{F}_{A_1,A_2}(B)$ is essentially $\mathcal{F}(H)$ and $\mathcal{F}(B)$ in Hilbert and abstract Wiener space settings, respectively, and also $I_a^{(1,-1)}[F(\cdot,\cdot)] = \mathcal{F}(F_0)$ where $F_0(x_1) = F(x_1, x_2)$ for all $(x_1, x_2) \in B \times B$. In this sense, Definition 2.1 includes the definition of $\mathcal{F}(H)$ and $\mathcal{F}(B)$ as special cases.

3. The analytic Feynman integral

In this section, we establish a translation theorem for the analytic Feynman integral of functions in $\mathcal{F}_{A_1,A_2}(B)$, and show how this integral can be modified so as to be translation invariant.

THEOREM 3.1. *Let $F \in \mathcal{F}_{A_1,A_2}(B)$ be given by (2.2). Then the analytic Feynman integral*

$$G(y_1, y_2) = I_a^q[F((\cdot,\cdot) + (y_1, y_2))] \tag{3.1}$$

exists for every $(y_1, y_2) \in B \times B$, and G also belongs to $\mathcal{F}_{A_1,A_2}(B)$.

Proof. Let the measure μ^* be defined by

$$\mu^*(E) = \int_E \exp\{i[(A_1^{\frac{1}{2}} h, y_1)^\sim + (A_2^{\frac{1}{2}} h, y_2)^\sim]\} d\mu(h)$$

for each $E \in \mathcal{B}(H)$. Then μ^* is in $M(H)$ and

$$F((x_1, x_2) + (y_1, y_2)) = \int_H \exp\{i[(A_1^{\frac{1}{2}} h, x_1)^\sim + (A_2^{\frac{1}{2}} h, x_2)^\sim]\} d\mu^*(h),$$

that is, $F((\cdot,\cdot)+(y_1, y_2))$ belongs to $\mathcal{F}_{A_1,A_2}(B)$. Therefore, by Theorem 2.3, $F((\cdot,\cdot) + (y_1, y_2))$ is analytic Feynman integrable and then

$$\begin{aligned} I_a^q[F((\cdot,\cdot) + (y_1, y_2))] &= \int_H \exp\{-\frac{i}{2}\sum_{k=1}^{2} q_k^{-1} < A_k h, h >\} d\mu^*(h) \\ &= \int_H \exp\{i[(A_1^{\frac{1}{2}} h, y_1)^\sim + (A_2^{\frac{1}{2}} h, y_2)^\sim]\} d\sigma(h) \end{aligned}$$

where the measure σ is defined by

$$\sigma(E) = \int_E \exp\{-\frac{i}{2}\sum_{k=1}^{2} q_k^{-1} < A_k h, h >\} d\mu(h)$$

for $E \in \mathcal{B}(H)$, and hence G is in $\mathcal{F}_{A_1,A_2}(B)$.

THEOREM 3.2. *Let $F \in \mathcal{F}_{A_1,A_2}(B)$ be given by (2.2) and let $y \in H$. Then*

$$\begin{aligned}
&\mathcal{I}_a^q[F((x_1,x_2) + (A_1^{\frac{1}{2}}y, A_2^{\frac{1}{2}}y))] \\
(3.2) \qquad &= \exp\{\frac{iq}{2}(< A_1y, y > + < A_2y, y >)\} \\
&\quad \mathcal{I}_a^q[F(x_1,x_2)\exp\{-iq[(A_1^{\frac{1}{2}}y, x_1)^{\sim} + (A_2^{\frac{1}{2}}y, x_2)^{\sim}]\}].
\end{aligned}$$

Proof. Since $F \in \mathcal{F}_{A_1,A_2}(B)$ is given by (2.2), we have

$$\begin{aligned}
&F(x_1,x_2)\exp\{-iq[(A_1^{\frac{1}{2}}y, x_1)^{\sim} + (A_2^{\frac{1}{2}}y, x_2)^{\sim}]\} \\
&\quad = \int_H \exp\{i[(A_1^{\frac{1}{2}}(h - qy), x_1)^{\sim} + (A_2^{\frac{1}{2}}(h - qy), x_2)^{\sim}]\}d\mu(h) \\
&\quad = \int_H \exp\{i[(A_1^{\frac{1}{2}}k, x_1)^{\sim} + (A_2^{\frac{1}{2}}k, x_2)^{\sim}]\}d\mu(k)
\end{aligned}$$

where $\mu(E) = \mu(E + qy)$ for $E \in \mathcal{B}(H)$. Using Theorem 2.3, we obtain

(3.3)

$$\begin{aligned}
&\mathcal{I}_a^q[F(x_1,x_2)\exp\{-iq[(A_1^{\frac{1}{2}}y, x_1)^{\sim} + (A_2^{\frac{1}{2}}y, x_2)^{\sim}]\}] \\
&= \int_H \exp\{-\frac{i}{2q}[< A_1(h - qy), h - qy > + < A_2(h - qy), h - qy >]\}d\mu(h) \\
&= \exp\{-\frac{iq}{2}[< A_1y, y > + < A_2y, y >]\} \int_H \exp\{-\frac{i}{2q}[< A_1h, h > \\
&\quad + < A_2h, h >]\}\exp\{i[< A_1h, y > + < A_2h, y >]\}d\mu(h).
\end{aligned}$$

By Theorem 3.1 and (3.3), the theorem is proved.

COROLLARY 3.3. *Let $F \in \mathcal{F}_{A_1,A_2}(B)$ be given by (2.2) and let $y \in H$. Then*

$$\begin{aligned}
\mathcal{I}_a^q[F(x_1,x_2)] &= \exp\{\frac{iq}{2}(< A_1y, y > + < A_2y, y >)\}\mathcal{I}_a^q[F((x_1,x_2) + \\
&\quad (A_1^{\frac{1}{2}}y, A_2^{\frac{1}{2}}y))\exp\{iq[(A_1^{\frac{1}{2}}y, x_1)^{\sim} + (A_2^{\frac{1}{2}}y, x_2)^{\sim}]\}].
\end{aligned}$$

REMARK 3.4. Let A_1 be the identity operator and $A_2 = 0$. Then (3.2) becomes the translation theorem for the analytic Feynman integral on abstract Wiener space, that is,

$$
\begin{aligned}
&\int_B^{anf_q} F(x+y)d\nu(x) \\
(3.4)\qquad &= \exp\{\frac{iq}{2}|y|^2\}\int_B^{anf_q} F(x)\exp\{-iq(y,x)^\sim\}d\nu(x)
\end{aligned}
$$

which is a well-known result in [7].

Next we introduce one particular Hilbert space [14,15] which has been used in the applications of the Fresnel integral to non-relativistic quantum mechanics Let H_N be the set of all functions $f : [0,1]^N \to R$ for which there exists v in $L_2([0,1]^N)$ such that

$$
(3.5)\qquad f(y_1,\cdots,y_N) = \int_0^{y_N}\cdots\int_0^{y_1} v(t_1,\cdots,t_N)dt_1\cdots dt_N
$$

for all $(y_1,\cdots,y_N) \in [0,1]^N$. And the inner product on H_N is defined by

$$
\begin{aligned}
&< f,g > \\
(3.6)\qquad &= \int_{[0,1]^N}(\frac{\partial^N f(t_1,\cdots,t_N)}{\partial t_1\cdots\partial t_N})(\frac{\partial^N g(t_1,\cdots,t_N)}{\partial t_1\cdots\partial t_N})dt_1\cdots dt_N.
\end{aligned}
$$

Then H_N is a separable Hilbert space. Also it will be helpful to introduce the family of functions $\{f_{(t_1,\cdots,t_N)} : (t_1,\cdots,t_N) \in [0,1]^N\}$ from H_N such that $f_{(t_1,\cdots,t_N)}(y_1,\cdots,y_N) = \prod_{k=1}^N \min\{t_k,y_k\}$ These functions have the reproducing property $< f, f_{(t_1,\cdots,t_N)} >= f(t_1,\cdots,t_N)$ for all $f \in H_N$. Further, it is well-known that (H_N, C_N, m_N) is an example of abstract Wiener space where $C_N \equiv C_N([0,1]^N)$ denotes N-parameter Wiener space with Wiener measure m_N. Also the above reproducing property carries over to the stochastic inner product (2 1) in the sense that $(x, f_{(t_1,\cdots,t_n)})^\sim = x(t_1,\cdots,t_N)$ for s-a,e. $x \in C_N$.

REMARK 3.5. Taking $(H,B,\nu) = (H_N, C_N, m_N)$, A_1 = the identity operator and $A_2 = 0$, we know that (3.2) becomes

$$\begin{aligned}(3.7)\quad \int_{C_N}^{anf_q} F(x+y)dm_N(x) = \exp\{\frac{iq}{2}\|\frac{\partial^N y}{\partial t_1\cdots\partial t_N}\|^2\}\int_{C_N}^{anf_q} F(x) \\ \exp\{-iq\int_{[0,1]^N}\frac{\partial^N y(t_1,\cdots,t_N)}{\partial t_1\cdots\partial t_N}dx(t_1,\cdots,t_N)\}dm_N(x)\end{aligned}$$

which contains Theorem 2 in [4] as a special case $N = 1$.

Finally we end this paper by showing how the analytic Feynman integral can be modified so as to be translation invariant.

Let $f : H \times H \to C$ be a measurable function such that the integral

$$K_f(\lambda_1, \lambda_2) = \int_{H\times H} f(\lambda_1^{-\frac{1}{2}}h_1, \lambda_2^{-\frac{1}{2}}h_2)d(m\times m)(h_1, h_2)$$

exists for all real $\lambda_1 > 0$ and $\lambda_2 > 0$. If there exists an analytic function $K_f^*(z)$ on $\Omega = \{z = (z_1, z_2) \in C^2 : Rez_k > 0 \quad \text{for} \quad k = 1, 2\}$ such that $K_f^*(\lambda_1, \lambda_2) = K_f(\lambda_1, \lambda_2)$ for all real $\lambda_1 > 0$ and $\lambda_2 > 0$, then we define $K_f^*(z)$ to be the analytic Gauss integral of f over $H \times H$ with parameter z, and for $z \in \Omega$ we write

$$I_a^z[f(\cdot,\cdot)] = K_f^*(z).$$

Let q_1 and q_2 be non-zero real numbers. If the limit

$$\lim_{z\to(-iq_1,-iq_2)} I_a^z[f(\cdot,\cdot)] = I_a^q[f(\cdot,\cdot)]$$

exists, $I_a^q[f(\cdot,\cdot)]$ is called the analytic Feynman integral of f over $H \times H$ with parameter $q = (q_1, q_2)$. In particular, if $q_1 = q_2 = q$, we write

$$\mathcal{I}_a^q[f(\cdot,\cdot)] = I_a^{(q,q)}[f(\cdot,\cdot)].$$

Using the above definition, the following theorems can be easily obtained from Theorem 2.3 and the same argument as it was used in Theorem 3.1 and Theorem 3.2.

THEOREM 3.6. *Let $f \in \mathcal{F}_{A_1,A_2}(H)$ be given by*

$$f(h_1,h_2) = \int_H \exp\{i[< A_1^{\frac{1}{2}}h, h_1 > + < A_2^{\frac{1}{2}}h, h_2 >]\}d\mu(h) \tag{3.8}$$

for some $\mu \in M(H)$. Then $I_a^q[f(\cdot,\cdot)]$ exists and

$$I_a^q[f(\cdot,\cdot)] = \int_H \exp\{-\frac{i}{2}\sum_{k=1}^{2} q_k^{-1} < A_k h, h >\}d\mu(h).$$

THEOREM 3.7. *Let $f \in \mathcal{F}_{A_1,A_2}(H)$ be given by (3.8) and let $y \in H$. Then $f((\cdot,\cdot) + (A_1^{\frac{1}{2}}y, A_2^{\frac{1}{2}}y)) \in \mathcal{F}_{A_1,A_2}(H)$ and*

$$\begin{aligned}\mathcal{I}_a^q[f((h_1,h_2) + (A_1^{\frac{1}{2}}y, A_2^{\frac{1}{2}}y))] &= \exp\{\frac{iq}{2}(< A_1y, y > + < A_2y, y >)\} \\ &\mathcal{I}_a^q[f(h_1,h_2)\exp\{-iq[< A_1^{\frac{1}{2}}y, h_1 > + < A_2^{\frac{1}{2}}y, h_2 >]\}].\end{aligned}$$

Let f be an analytic Feynman integrable function over $H \times H$ for some non-zero real q. Let

$$g(h_1,h_2) = \exp\{\frac{iq}{2}[|h_1|^2 + |h_2|^2]\}f(h_1,h_2) \tag{3 9}$$

for $(h_1,h_2) \in H \times H$. Then we define the translation invariant analytic Feynman integral of g with parameter q by

$$\mathcal{I}_a^{ti_q}[g(\cdot,\cdot)] = \mathcal{I}_a^q[f(\cdot,\cdot)] \tag{3.10}$$

THEOREM 3.8. *Let $f \in \mathcal{F}_{A_1,A_2}(H)$ and g be given by (3.8) and (3,9) respectively. Then, for each $y \in H$, $g((\cdot,\cdot)+(A_1^{\frac{1}{2}}y, A_2^{\frac{1}{2}}y))$ is translation invariant analytic Feynman integrable over $H \times H$ with parameter q, and*

$$\mathcal{I}_a^{ti_q}[g((\cdot,\cdot) + (A_1^{\frac{1}{2}}y, A_2^{\frac{1}{2}}y))] = \mathcal{I}_a^{ti_q}[g(\cdot,\cdot)]. \tag{3.11}$$

Proof. By (3.9), we have, for $y \in H$,

$$\begin{aligned}
&g((\cdot,\cdot)+(A_1^{\frac{1}{2}}y, A_2^{\frac{1}{2}}y))\\
&= \exp\{\frac{iq}{2}[< h_1, h_1 > + < h_2, h_2 >]\}\exp\{-iq[< A_1^{\frac{1}{2}}y, h_1 > + < A_2^{\frac{1}{2}}y,\\
&\quad h_2 >]\}\exp\{\frac{iq}{2}[< A_1 y, y > + < A_2 y, y >]\}f((h_1,h_2)+(A_1^{\frac{1}{2}}y, A_2^{\frac{1}{2}}y)).
\end{aligned}$$

Since $f((h_1,h_2)+(A_1^{\frac{1}{2}}y, A_2^{\frac{1}{2}}y)) \in \mathcal{F}_{A_1,A_2}(H)$, we know that

$$f((h_1,h_2)+(A_1^{\frac{1}{2}}y, A_2^{\frac{1}{2}}y)) = \int_H \exp\{i[< A_1^{\frac{1}{2}}k, h_1 > + < A_2^{\frac{1}{2}}k, h_2 >]\}d\mu(k)$$

for some $\mu \in M(H)$. Then we can write

$$\begin{aligned}
&g((h_1,h_2)+(A_1^{\frac{1}{2}}y, A_2^{\frac{1}{2}}y))\\
&= \exp\{\frac{iq}{2}[|h_1|^2+|h_2|^2]\}\int_H \exp\{i[< A_1^{\frac{1}{2}}h, h_1 > + < A_2^{\frac{1}{2}}h, h_2 >]\}d\mu(h)
\end{aligned}$$

where $\mu(E) = \mu^*(E+qy)$ and $\mu^*(E) = \exp\{\frac{iq}{2}[< A_1 y, y > + < A_2 y,$ $y >]\}\int_H \exp\{i[< A_1^{\frac{1}{2}}k, h_1 > + < A_2^{\frac{1}{2}}k, h_2 >]\}d\mu(k)$ for $E \in \mathcal{B}(H)$. Hence $g((h_1,h_2)+(A_1^{\frac{1}{2}}y, A_2^{\frac{1}{2}}y))$ is translation invariant analytic Feynman integrable.

Applying Theorem 3 7, (3.9) and (3.10), we obtain

$$\begin{aligned}
&\mathcal{I}_a^{t_2 q}[g((h_1,h_2)+(A_1^{\frac{1}{2}}y, A_2^{\frac{1}{2}}y))]\\
&= \mathcal{I}_a^q[\exp\{iq[< A_1^{\frac{1}{2}}y, h_1 > + < A_2^{\frac{1}{2}}y, h_2 >]\}\exp\{\frac{iq}{2}[< A_1 y, y >\\
&\quad + < A_2 y, y >]\}f((h_1,h_2)+(A_1^{\frac{1}{2}}y, A_2^{\frac{1}{2}}y))]\\
&= \exp\{\frac{iq}{2}[< A_1 y, y > + < A_2 y, y >]\}\mathcal{I}_a^q[f((h_1,h_2)+(A_1^{\frac{1}{2}}y, A_2^{\frac{1}{2}}y))\\
&\quad \exp\{iq[< A_1^{\frac{1}{2}}y, h_1 > + < A_2^{\frac{1}{2}}y, h_2 >]\}]\\
&= \mathcal{I}_a^q[f(\cdot,\cdot)] = \mathcal{I}_a^{t_2 q}[g(\cdot,\cdot)].
\end{aligned}$$

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