East Asian Math. J. **14** (1998), No. 2, pp 227–235

APPROXIMATE FIBRATIONS AND APPROXIMATE LOCALLY TRIVIAL BUNDLES

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1. Introduction

Until now, mathematicians have used many properties or concepts of cell-like mappings and Hurewicz fibrations for the research of the upper semicontinuous decompositions of manifolds or mappings between manifolds. But, the preimage of each point of cell-like mapping has a trivial shape, and so it is limited in some sense. Thus, we need to introduce the concept of approximate fibrations[1] which have the approximate homotopy lifting property for every topological space.

Y. H. Im[4] investigated conditions under which approximate fibrations can be approximated by locally trivial bundle.

In this paper, we newly define a generalized concept of locally trivial bundles, called an approximate locally trivial bundle, and show that it induces an approximate fibration as a locally trivial bundle implies a fibration.

2. Preliminaries

We use the following terminology and notation. If $H : X \times I \to Y$ is a homotopy, then $H_t : X \to Y$ is the map defined by $H_t(x) = H(x, t)$ Let $f : X \to Y$ and $g : X \to Y$ be maps and δ be a cover of Y. We say that f and g are δ-close if for each $x \in X$, $f(x)$ and $g(x)$ are contained in some member of δ Also f and g are δ-homotopic if f and g are homotopic by a homotopy h such that $h(\{x\} \times I)$ is contained in some member of δ for each $x \in X$. Such a homotopy is called a δ-homotopy[3].

Received February 3, 1998 Revised August 23, 1998

DEFINITION 2.1. A surjective map $p : E \to B$ between locally compact ANRs has the approximate homotopy lifting property with respect to a space X provided that, given a cover ϵ of B and maps $g : X \to E$ and $H : X \times I \to B$ such that $pg = H_0$, there exists a map $G : X \times I \to E$ such that $G_0 = g$ and pG and H are ϵ-close. The map G is said to be an ϵ-lift of H. If p has the AHLP (approximate homotopy lifting property) for all spaces, we say that p is an approximate fibration.

A fiber structure (E, p, B) is called a fibration for class $\mathcal{A}$ if, for any space $X \in \mathcal{A}$, each continuous $f : X \times 0 \to E$ and each homotopy $\phi : X \times I \to B$ of pf, there exists a homotopy $\tilde{\phi}$ of f covering ϕ, that is, $p\tilde{\phi} = \phi$. A fiber space for the class of all spaces is called a Hurewicz fibration. A map $p : E \to B$ is a weak approximate fibration if p satisfies the AHLP for I^q for all $q < \infty$. In [2], p is a weak approximate fibration if and only if it is an approximate fibration.

The above Definition 2 1, of course, generalizes the usual homotopy lifting property, the definition of which is the same except that $pG = H$ is required rather than that pG and H are ϵ-close.

DEFINITION 2.2. A map $p : E \to B$ is a locally trivial bundle between topological spaces if for each $b \in B$ there exist a neighborhood V of b in B and a homeomorphism ϕ_V of $V \times p^{-1}(b)$ onto $p^{-1}(V)$ such that $p\phi_V(v, x) = v$ for all $(v, x) \in V \times p^{-1}(b)$.

In a locally trivial bundle, all fibers (i.e., all subspaces of E of the form $p^{-1}(b)$) are homeomorphic.

We give a new notion as above so that we can apply for more general cases.

DEFINITION 2.3. A map $p : E \to B$ is an approximate locally trivial bundle between locally compact ANRs if given a cover ϵ of B and $b \in B$ there exist a neighborhood V of b in B and a homeomorphism ϕ_V of $p^{-1}(b) \times V$ onto $p^{-1}(V)$ such that $p\phi_V(v, x)$ and $\alpha_V(v, x)$ are ϵ-close for all $(v, x) \in V \times p^{-1}(b)$, where $\alpha_V : V \times p^{-1}(b) \to V$ is the first projection.

3. Main Results

In [5, p 364] and [6, p 96], a locally trivial bundle $p : E \to B$ with a fiber F is a weak fibration By using the homotopy extension property of ANR's, we extend this fact to the case of approximate locally trivial bundles.

THEOREM 3.1. *An approximate locally trivial bundle $p : E \to B$ with fiber $F = p^{-1}(b)$ is an (weak) approximate fibration.*

Proof. Step 1. Let $p : E \to B$ be an approximate locally trivial bundle with fiber F. Then, given any cover ϵ of B and $b \in B$, there are a neighborhood V of b in B and a homeomorphism $\phi_V . V \times F \to p^{-1}(V)$ such that $p\phi_V(v,x)$, $\alpha_V(v,x)$ are ϵ-close for all $(v,x) \in V \times F$, where $\alpha_V : V \times F \to V$ is the first projection Consider the commutative diagram

$$\begin{array}{ccc} I^n & \xrightarrow{\tilde{f}} & E \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ I^n \times I & \xrightarrow[G]{} & B, \end{array}$$

where $i : I^n \to I^n \times I$ is defined by $i(x) = (x, 0)$.
We claim that there exists a map $\tilde{G} . I^n \times I \to E$ such that $\tilde{G}(x,0) = \tilde{f}(x)$ and $p\tilde{G}$ and G are ϵ-close.

$$\text{Let } \delta = \{G^{-1}(V) | \text{each coordinate neighborhood } V \text{ of bundle}\}$$

be an open cover of the compact metric space $I^n \times I$.
If λ is the Lebesque number of the cover δ, then any subset $A \subset I^n \times I$ with $diam(A) < \lambda$ implies that $G(A) \subset V$ for some $V \in \epsilon$. Triangulate I^n so that every simplex σ of I^n has $diam(\sigma) < \lambda/2$. Partition $I : 0 = t_0 < t_1 < t_2 < \cdots < t_m = 1$ so that $diam[t_j, t_{j+1}] = t_{j+1} - t_j < \lambda/2$ for $0 \leq j \leq m$ Then $diam(\sigma \times [t_j, t_{j+1}]) < 1/2\lambda + 1/2\lambda = \lambda$ for each σ and j. Hence, for each σ, j, there is a coordinate neighborhood $V = V_{\sigma,j}$ such that $G(\sigma \times [t_j, t_{j+1}]) \subset V$.
Step 2. Let L denote the simplicial complex of the triangulation of I^n, and let $L^{(k)}$ denote k-skeleton of L.

We show that, by induction on $k \geq 0$, there is a continuous map $\tilde{h}_k : L^{(k)} \times [0, t_1] \to E$ such that

$$\begin{array}{ccc} L^{(k)} & \xrightarrow{\tilde{f}} & E \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ L^{(k)} \times [0, t_1] & \xrightarrow[G]{} & B \end{array}$$

and $\tilde{h}_k : L^{(k)} \times [0, t_1] \to E$ is ϵ-lift of G.

Let $\alpha_V : V \times F \to V$, $\beta_V : V \times F \to F$ be the projections. If $e \in p^{-1}(V)$, then

$$\phi_V^{-1}(e) = (\alpha_V \phi_V^{-1}(e), \beta_V \phi_V^{-1}(e)) \in V \times F,$$

and hence $\phi_V^{-1}(e) = (v, x)$ for some $(v, x) \in V \times F$; $\phi_V(v, x) = e$. Since $p\phi_V(v, x), \alpha_V(v, x)$ are ϵ-closed, $p(e), \alpha_V \phi_V^{-1}(e)$ are also ϵ-closed.

(1) If $\mu \in L^{(0)}$, then there is a coordinate neighborhood V such that $G(\{\mu\} \times [0, t_1]) \subset V$. Define homotopy $\tilde{h}'_0 : \{\mu\} \times [t_1/2, t_1] \to p^{-1}(V)$ by $\tilde{h}'_0(\mu, t) = \phi_V(G(\mu, t), \beta_V \phi_V^{-1} \tilde{f}(\mu))$. Since B is an ANR, by definition, given a cover ϵ of B, there is a cover δ such that any two δ-close maps into are η-homotopic, where η twice star refines ϵ. Hence we can define homotopy $H : \{\mu\} \times [0, t_1/2] \to p^{-1}(V)$ by $H(\mu, 0) = \tilde{f}(\mu), H(\mu, t_1/2) = \tilde{h}'_0(\mu, t_1/2)$. Define $\tilde{h}_0 : \{\mu\} \times [0, t_1] \to p^{-1}(V)$ by

$$\tilde{h}_0(\mu, t) = \begin{cases} H(\mu, t) & (0 \leq t \leq t_1/2) \\ \tilde{h}_0{}'(\mu, t) & (t_1/2 \leq t \leq t_1). \end{cases}$$

Thus $\tilde{h}_0(\mu, 0) = \tilde{f}(\mu)$ and $p\tilde{h}_0(\mu, t)$, $G(\mu, t)$ are ϵ-close. Since $L^{(0)}$ is discrete, one may glue these map $\tilde{h}_0$ together to obtain a continuous map $\tilde{h}_0 : L^{(0)} \times [0, t_1] \to E$.

(2) Suppose that for $k-1$, there is a continuous map $\tilde{h}_{k-1} : L^{(k-1)} \times [0, t_1] \to E$ such that

$$\begin{array}{ccc} L^{(k-1)} & \xrightarrow{\tilde{f}} & E \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle p} \\ L^{(k-1)} \times [0, t_1] & \xrightarrow[G]{} & B \end{array}$$

and $\tilde{h}_{k-1} : L^{(k-1)} \times [0, t_1] \to E$ is ϵ-lift of G.
(3) Let σ be a k-simplex in L, and that let V be a coordinate neighborhood such that $G(\sigma \times [0,t_1]) \subset V$. Since $\sigma \cong I^k$, a retraction $r : I^k \times I \to (I^k \times \{0\}) \cup (\dot{I}^k \times I)$ induce a retraction $r_\sigma : \sigma \times [0, t_1] \to \sigma \times \{0\}) \cup (\dot{\sigma} \times [0, t_1])$. Define

$$\tilde{V_\sigma} : (\sigma \times \{0\}) \cup (\dot{\sigma} \times [0, t_1]) \to p^{-1}(V)$$

by $\tilde{V}_{\sigma|\sigma \times \{0\}} = \tilde{f} i_{|\sigma \times \{0\}}$ and $\tilde{V}_{\sigma|\dot{\sigma} \times [0,t_1]} = \tilde{h}_{k-1_{|\dot{\sigma} \times [0,t_1]}}$ by induction. Define

$$\tilde{h'_k} : \sigma \times [t_1/2, t_1] \to p^{-1}(V)$$

by $\tilde{h'_k}(\mu, t) = \phi_V(G(\mu, t), \beta_V \phi_V^{-1} \tilde{V}_\sigma r_\sigma(\mu, t))$ for $\mu \in \sigma \subset L^{(k)}, t \in [0, t_1]$. Similarly, there is $H : \sigma \times [0, t_1/2] \to p^{-1}(V)$ by $H(\sigma, 0) = \tilde{f}(\mu)$ and $H(\sigma, t_1/2) = \tilde{h}'_k(\mu, t_1/2)$. Define $\tilde{h_k} : \sigma \times [0, t_1] \to p^{-1}(V)$ by

$$\tilde{h_k}(\sigma, t) = \begin{cases} H(\sigma, t) & (0 \le t \le t_1/2) \\ \tilde{h}'_k & (t_1/2 \le t \le t_1) \end{cases}$$

If $(\mu, t) \in (\sigma \times \{0\}) \cup (\dot{\sigma} \times [0, t_1])$, then $r_\sigma(\mu, t) = (\mu, t)$ by retraction. If $(\mu, t) \in \sigma \times \{0\}$, then $G(\mu, t) = G(\mu, 0) = p\tilde{f}(\mu)$ and $\tilde{V}_\sigma(\mu, 0) = \tilde{f}(\mu)$, hence $\tilde{h_k}(\mu, 0) = \tilde{f}(\mu)$. If $(\mu, t) \in \dot{\sigma} \times [0, t_1]$, then $G(\mu, t), p\tilde{h}_{k-1}$ are ϵ-close by induction and $\tilde{V}_\sigma(\mu, t) = \tilde{h}_{k-1}(\mu, t)$. Hence $p\tilde{h}_k(\mu, t), G(\mu, t)$ are ϵ-close. Since simplexes in L intersect in lower dimension faces, the gluing lemma allow us to assemble all the map $\tilde{h_k} : L^{(k)} \times [0, t_1] \to E$.
Step 3. In particular $k = n$, there is a continuous map $\tilde{G_1} = \tilde{h_n}$: $I^n \times [0, t_1] \to E$. Now repeat this construction with $[t_1, t_2]$ playing the role of $[t_0, t_1]$ to obtain a map $I^n \times [t_1, t_2] \to E$ agree with $\tilde{G}_1$ on $I^n \times \{t_1\}$. These maps can be glued together to obtain a map $\tilde{G}_2$: $I^n \times [0, t_2] \to E$ making the appropriate diagram commute. Repeating, we obtain $\tilde{G} = \tilde{G_m}$ defined on $I^n \times [0, t_m] = I^n \times I$

REMARK. In general, the set

$$\mathcal{B}(E,B) = \{p : E \to B | p \text{ is a locally trivial bundle}\}$$

is not closed in $\mathcal{C}(E,B) = \{f : E \to B|\ f \text{ is continuous }\}$ with compact open topology but $\mathcal{AB}(E,B) = \{p : E \to B|\ p \text{ is an approximate locally trivial bundle }\}$ is closed in $\mathcal{C}(E,B)$.

The following result follows from [1, proposition 1.1.].

THEOREM 3.2 [1]. *Let E and B be ANR'S. Suppose $p : E \to B$ is a surjection with the property that for each cover δ of B there is a map $p_\delta : E \to B$ such that p_δ is δ-close to p and p_δ has the homotopy lifting property with respect to a metric space X. Then p has the approximate homotopy lifting property with respect to X.*

However, the above result does not hold for an approximate locally trivial bundles. The following is a counterexample.

EXAMPLE. Let W be the Warsaw circle in R^2 ; that is, $W = W_1 \cup B$, where

$$W_1 = \{(0,t)| -1 \le t \le 1\} \cup \{(x, sin(\pi/x)\}$$

and B is an arc which meets W_1 only in its endpoints $(0,0)$ and $(1,0)$. That is, let x_0 be a base point in S^1. Let $\pi : S^1 \times S^1 \to S^1$ be projection map onto second factor. Then there is a compactum $A \subset S^1 \times S^1$ such that $A \cong W$ and $h : (S^1 \times S^1) - A \xrightarrow{\cong} S^1 \times (S^1 - \{x_0\})$. Define $p : S^1 \times S^1 \to S^1$ by

$$p(x) = \begin{cases} \pi h(x) & \text{if } x \in (S^1 \times S^1) - A \\ x_0 & \text{if } x \in A. \end{cases}$$

Then p is continuous and $p^{-1}(x_0) = A$, and $p^{-1}(y)$ essential copy of S^1 for all $y \ne x_0$. Now p can be uniformly approximated by locally trivial bundle. In fact, let $\delta > 0$ be given and let U be an open interval in S^1 such that $x_0 \in U \subset N(x_0, \delta/2)$. Then, since $p^{-1}(S^1 - U)$ and $p^{-1}(cl(U)) \cong S^1 \times I$, $p_{|p^{-1}(S^1-U)}$ extends to a map $p_\delta : S^1 \times S^1 \to S^1$ which topological equivalent to π. Thus p_δ is locally trivial bundle and p_δ, p are δ-close. However, p is not an approximate locally trivial bundle since $p^{-1}(x_0)$ is not homeomorphic to $p^{-1}(y)$, $y \ne x_0$.

From the above example, approximate fibrations can not be in general approximate locally trivial bundles. We investigate some conditions under which an approximate fibrations $p : E \to B$ can be an approximate locally trivial bundle.

Let F^m be a compact manifold. We denote by $S(F)$ the set of equivalence classes of the form [f], where $f : M^m \to F^m$ is a homotopy equivalence of a compact manifold M^m to F^m which is a homeomorphism from ∂M to ∂F.

If T^n is the n-torus and $e : T^n \to T^n$ is any standard finite cover, then there is a transfer map $\hat{e} : S(T^n \times F) \to S(T^n \times F)$ defined by $\hat{e}([f]) = [\hat{f}]$, where $\hat{f}$ comes from the pull-back diagram

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\hat{f}} & T^n \times F \\ \downarrow & & \downarrow{\scriptstyle e\times id} \\ M & \xrightarrow{f} & T^n \times F. \end{array}$$

We use $S_0(T^n \times F)$ to denote those elements of $S(T^n \times F)$ that are invariant under any of these transfer maps([4]).

LEMMA 3.3 [4]. *Let $n \geq 0$ be an integer. For any $\epsilon > 0$, there exists a $\delta > 0$ so that if $f : M^{m+n} \to R^n \times F^m$ is a $p^{-1}(\delta)$-equivalence for which $f_{|\partial M} : \partial M \to R^n \times \partial F^m$ is a homeomorphism, where M^{m+n} is a manifold , F^m is a compact manifold with boundary and $m + n \geq 5$, then there is an element $\sigma(f)$ of $S_0(T^m \times F)$ which vanishes if and only if f is $p^{-1}(\epsilon)$-homotopic to a homeomorphism.*

THEOREM 3.4. *Suppose that $p : M^{m+n} \to B^n$ is a surjection with the property that for each cover δ of B there is a map $p_\delta : M \to B$ such that p_δ is δ-close to p and p_δ is the locally trivial bundle and $S(T^m \times F) = 0$ for each fiber F. Then p is an approximate locally trivial bundle.*

Proof. Assume that for all $\delta \geq 0$ and $b \in B$, there is a neighborhood V of b and a homeomorphism $\phi_V : V \times F \xrightarrow{\cong} p_\delta^{-1}(V)$ such that $p_\delta \phi_V(v, x) = \alpha_V(v) = v$, where $\alpha_V : V \times F \to V$ and for all $(v, x) \in V \times F$. Let p_δ and p be δ-close. By [4], the map $\phi_V : V \times F \to p^{-1}(V)$ is homotopy equivalence. Since $S(T^m \times F) = 0$, $V \times F \cong p^{-1}(V)$. Thus, since p_δ and p are δ-close, $p\phi_V(v, x)$ and $p_\delta \phi_V(v, x)$ are δ-close. Since

$p_\delta\phi_V(v,x) = \alpha_V(v,x)$, $p\phi_V(v,x)$ and $\alpha_V(v,x)$ are δ-close. Therefore $p : M \to B$ is an approximate locally trivial bundle.

The next are some examples satisfying the condition of Theorem 3.4.

EXAMPLE 1. It follows from [4] that if F^m is a $K(\pi,1)$ with π poly Z and $m+n \geq 5$, then $S(T^n \times F^m) = 0$. Also any sphere S^m satisfies $S(T^n \times S^m) = 0$.

EXAMPLE 2. Let F^m be a closed Riemannian manifold whose sectional curvature value is nonpositive. Then, by Proposition 3.5 in [4], $S(F^m \times I^j) = 0$ for $m+j \geq 5$.

CROLLARY 3.5. *Suppose $p : M^{m+n} \to B^n$ is an approximate fibration whose fiber is homotopy equivalent to a closed manifold F^m and $m+n \geq 5$, where either F^m is a $K(\pi,1)$ with π poly Z or a closed Riemannian manifold whose sectional curvature is nonpositive. Then p is an approximate locally trivial bundle.*

Proof. By [4, Theorem 4.9] and examples, p can be approximated by a locally trivial bundle. Hence the result follows from Theorem 3.4.

ACKNOWLEDGEMENT. *The authors are deeply grateful to the refree for the valuable suggestions.*

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