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INTERPOLATION SPACES GENERATED BY ANALYTIC SEMIGROUP OPERATORS

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1. Introduction

In this paper we consider an interpolation method between the initial Banach space and the domain of the infinitesimal generator A of the analytic semigroup $T(t)$, and the fundamental results of the corresponding theorems in the new setting. The objects are obtained by the development of an interpolation theory between Banach spaces X and Y, which is denoted by $(X, Y)_{\theta,p}$, in particular by the J- and K-methods as in Butzer and Berens [1] and [2] We will verify the fact that

$$(D(A), X)_{\theta,p} = \left\{ x \in X : \int_0^t (t^\theta ||AT(t)x||)^p \frac{dt}{t} < \infty \right\},$$

for $0 < \theta < 1$, $1 \leq p \leq \infty$.

It is mainly on the role of interpolation spaces in the study of analytic semigroup of operators. In forth coming paper, we will deal with interpolation spaces between the initial Banach spaces and the domain of the regularity dissipative operator.

2. Preliminaries

Let X and Y be two Banach spaces contained in a locally convex linear Hausdorff space $\mathcal{X}$ such that the embedding mapping of both X and Y in $\mathcal{X}$ is continuous. Let $X \cap Y$ be a dense subspace in both X and Y. For $1 < p < \infty$, we denote by $L^p_*(X)$ the Banach space of all functions $t \to u(t)$, $t \in (0, \infty)$ and $u(t) \in X$, for which the mapping

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$t \to u(t)$ is strongly measurable with respect to the measure dt/t and the norm $||u||_{L^p_*(X)}$ is finite, where

$$||u||_{L^p_*(X)} = \{\int_0^\infty ||u(t)||^p_X \frac{dt}{t}\}^{\frac{1}{p}}.$$

For $0 < \theta < 1$, set

$$||t^\theta u||_{L^p_*(X)} = \{\int_0^\infty ||t^\theta u(t)||^p_X \frac{dt}{t}\}^{\frac{1}{p}},$$
$$||t^\theta u'||_{L^p_*(Y)} = \{\int_0^\infty ||t^\theta u'(t)||^p_Y \frac{dt}{t}\}^{\frac{1}{p}}.$$

We now introduce a Banach space

$$V = \{u : ||t^\theta u||_{L^p_*(X)} < \infty, \quad ||t^\theta u'||_{L^p_*(Y)} < \infty\}$$

with norm

$$||u||_V = ||t^\theta u||_{L^p_*(X)} + ||t^\theta u'||_{L^p_*(Y)}.$$

DEFINITION 2.1. We define $(X,Y)_{\theta,p}$, $0<\theta<1$, $1 \le p \le \infty$, to be the space of all elements $u(0)$ where $u \in V$, that is,

$$(X,Y)_{\theta,p} = \{u(0) : u \in V\}.$$

For $0<\theta<1$ and $1 \le p \le \infty$, the space $(X,Y)_{\theta,p}$ is a Banach space with the norm

$$||a||_{\theta,p} = \inf\{||u|| : u \in V, \quad u(0) = a\}.$$

Furthermore, there is a constant $C_\theta > 0$ such that

$$||a||_{\theta,p} = C_\theta \inf\{||t^\theta u||^{1-\theta}_{L^p_*(X)} ||t^\theta u'||^{\theta}_{L^p_*(Y)} : u(0) = a, \quad u \in V\}.$$

as is seen in [2]. It is known that $(X,X)_{\theta,p} = X$ for $0 < \theta < 1$ and $1 \le p \le \infty$ and

$$(X,Y)_{\theta,p} \subset (X,Y)_{\theta',p}, \quad 0 < \theta < \theta' < 1$$

where $X \subset Y$ satisfying that there exists a constant $c > 0$ such that $||u||_Y \leq c||u||_X$.

Let X_1 and Y_1 [resp. X_2 and Y_2] be two Banach spaces contained in a locally convex linear Hausdorff space $\mathcal{X}_1$ [resp. $\mathcal{X}_2$] such that the embedding mappings of both X_1 and Y_1 [both X_2 and Y_2] in $\mathcal{X}_1$ [resp $\mathcal{X}_2$] are continuous. Let $T : \mathcal{X}_1 \to \mathcal{X}_2$ be linear operator such that $T \in B(X_1, X_2)$ and $T \in B(Y_1, Y_2)$ where $B(X, Y)$ denotes the space of all bounded linear operators. Then the following result is obtained from Theorem 3.1 in [2].

PROPOSITION 2 1. *If* $T \in B(X_1, X_2) \cap B(Y_1, Y_2)$, *then* $T \in B((X_1, Y_1)_{\theta,p}, (X_2, Y_2)_{\theta,p})$ *satisfying*

$$||T||_{B((X_1,Y_1)_{\theta,p},(X_2,Y_2)_{\theta,p})} \leq ||T||^{1-\theta}_{B(X_1,X_2)}||T||_{B(Y_1,Y_2)}.$$

Let X be a Banach space with norm $||\cdot||$ and $T(t)$ be a C_0-semigroup with infinitesimal generator A. Then its domain $D(A)$ is a Banach space with the graph norm $||x||_{D(A)} = ||Ax|| + ||x||$.

The following result is obtained from Theorem 3.1 in [3].

PROPOSITION 2.2. *Let* $0 < \theta < 1$, $1 < p < \infty$. *Then*

$$(D(A), X)_{\theta,p} = \{x \in X : \int_0^t (t^{\theta-1}||T(t)x - x||)^p \frac{dt}{t} < \infty\}.$$

In particular. let $p = \frac{1}{\theta}$. Then

$$||t^\theta u||_{L^p_*(X)} = \{\int_0^\infty ||t^\theta u(t)||^p_X \frac{dt}{t}\}^{\frac{1}{p}} = \{\int_0^\infty ||u(t)||^{\frac{1}{\theta}}_X dt\}^\theta,$$

$$||t^\theta u'||_{L^p_*(Y)} = \{\int_0^\infty ||u'(t)||^{\frac{1}{\theta}}_Y dt\}^\theta,$$

and hence

$$V = \{u : u \in L^{\frac{1}{\theta}}(0, \infty; X), \quad u' \in L^{\frac{1}{\theta}}(0, \infty; Y)\}.$$

THEOREM 2.1. *Let* X *be a dense subspace of* Y *and the embedding mapping is continuous. Then*

$$V = L^{\frac{1}{\theta}}(0, \infty; X) \cap W^{1,\frac{1}{\theta}}(0, \infty; Y) \subset C([0, \infty); (X, Y)_{\theta,\frac{1}{\theta}})$$

Proof. If $u \in V$ then

$$||u||_V = \{\int_0^\infty ||u(t)||_X^{\frac{1}{\theta}} dt\}^\theta + \{\int_0^\infty ||u'(t)||_Y^{\frac{1}{\theta}} dt\}^\theta.$$

Putting $v(t) = u(t+\tau)$ for every $u \in V$ and $\tau > 0$, we see

$$v \in V, \quad u(\tau) = v(0) \in (X, Y)_{\theta, \frac{1}{\theta}}.$$

For $0 \le \sigma < \tau < \infty$, it holds

$$\begin{aligned} ||u(\tau) - u(\sigma)||_{\theta, \frac{1}{\theta}} &\le ||u(\cdot + \tau) - u(\cdot + \sigma)||_V \\ &\le \{\int_0^\infty ||u(t+\tau) - u(t+\sigma)||_X^{\frac{1}{\theta}} dt\}^\theta \\ &\quad + \{\int_0^\infty ||u'(t+\tau) - u(t+\sigma)||_Y^{\frac{1}{\theta}} dt\}^\theta \\ &\to 0 \end{aligned}$$

as $\tau - \sigma \to 0$. Therefore the mapping

$$[0, \infty) \ni \tau \mapsto u(\tau) \in (X, Y)_{\theta, \frac{1}{\theta}}$$

is continuous.

COROLLARY 2.1. *Let $T(t)$ be a C_0-semigroup with generator A in X. Then*

$$L^{\frac{1}{\theta}}(0, \infty; D(A)) \cap W^{1, \frac{1}{\theta}}(0, \infty; X) \subset C([0, \infty); (D(A), X)_{\theta, \frac{1}{\theta}}).$$

3. Main results

Let X be a Banach space with norm $||\cdot||$ and $T(t)$ be an analytic semigroup with infinitesimal generator A. We may assume that

$$||T(t)|| \le M, \quad ||AT(t)|| \le \frac{K}{t}$$

for some positive constants M, K and $t \ge 0$.

The proof of following Lemma is from Lemma 2.1 in [3].

LEMMA 2.1. *Let $0 < \theta < 1$, $1 < p < \infty$ and $\phi(t) \geq 0$ almost everywhere. Then*

$$\{\int_0^\infty (t^{\theta-1}\int_0^t \phi(s)ds)^p \frac{dt}{t}\}^{\frac{1}{p}} \leq \frac{1}{1-\theta}\{\int_0^\infty (t^\theta \phi(t))^p \frac{dt}{t}\}^{\frac{1}{p}}.$$

The main result in this paper is the following.

THEOREM 3.1. *Let $0 < \theta < 1$, $0 \leq t$. Then*

$$\begin{aligned}(1-\theta)\{&\int_0^\infty (t^{\theta-1}\|(T(t)-I)x\|)^p \frac{dt}{t}\}^{\frac{1}{p}} \\ \leq \{&\int_0^\infty (t^\theta\|AT(t)x\|)^p \frac{dt}{t}\}^{\frac{1}{p}} \\ \leq \frac{K}{1-2^{-\theta}}\{&\int_0^\infty (t^{\theta-1}\|(T(t)-I)x\|)^p \frac{dt}{t}\}^{\frac{1}{p}}.\end{aligned}$$

Therefore, we have

$$(D(A),X)_{\theta,p} = \{x \in X : \int_0^\infty (t^\theta\|AT(t)x\|)^p \frac{dt}{t} < \infty\}.$$

Proof. From

$$\begin{aligned}AT(t)x &= -\sum_{k=0}^{n}(AT(2^{k+1}t) - AT(2^k t))x + AT(2^{n+1}t)x \\ &= -\sum_{k=0}^{n}(AT(2^k t)(T(2^k t) - I)x + AT(2^{n+1}t)x\end{aligned}$$

it follows that

$$\|AT(t)x\| \leq \sum_{k=0}^{n}\frac{K}{2^k t}\|(T(2^k t)-I)x\| + \frac{K}{2^{n+1}t}\|x\|,$$

$$\begin{aligned}t^\theta\|AT(t)x\| \leq K&\sum_{k=0}^{n} 2^{-k\theta}(2^k t)^{\theta-1}\|(T(2^k t)-I)x\| \\ &+ \frac{K}{2^{n+1}}\|x\|t^{\theta-1},\end{aligned}$$

and hence

$$\begin{aligned}
&\{\int_0^\infty (t^\theta ||AT(t)x||)^p \frac{dt}{t}\}^{\frac{1}{p}} \\
&\leq K\sum_{k=0}^{n} 2^{-k\theta}\{\int_\epsilon^\infty ((2^k t)^{\theta-1}||(T(2^k t)-I)x||)^p \frac{dt}{t}\}^{\frac{1}{p}} \\
&\quad + \frac{K}{2^{n+1}}||x||\{\int_\epsilon^\infty t^{(\theta-1)p}\frac{dt}{t}\}^{\frac{1}{p}} \\
&= K\sum_{k=0}^{n} 2^{-k\theta}\{\int_{2^k\epsilon}^\infty (t^{\theta-1}||(T(t)-I)x||)^p \frac{dt}{t}\}^{\frac{1}{p}} \\
&\quad + \frac{K}{2^{n+1}}||x||(\frac{\epsilon^{(\theta-1)p}}{(1-\theta)p})^{\frac{1}{p}}, \quad (2^k t \to t),
\end{aligned}$$

for every $\epsilon > 0$. Thus,

$$\begin{aligned}
&\{\int_\epsilon^\infty (t^\theta ||AT(t)x||)^p \frac{dt}{t}\}^{\frac{1}{p}} \\
&\leq K\sum_{k=0}^{\infty} 2^{-k\theta}\{\int_{2^k\epsilon}^\infty (t^{\theta-1}||(T(t)-I)x||)^p \frac{dt}{t}\}^{\frac{1}{p}}
\end{aligned}$$

as $n \to \infty$. Therefore, passing $\epsilon \to 0$, we obtain

$$\begin{aligned}
&\{\int_0^\infty (t^\theta ||AT(t)x||)^p \frac{dt}{t}\}^{\frac{1}{p}} \\
&\leq K\sum_{k=0}^{\infty} 2^{-k\theta}\{\int_0^\infty (t^{\theta-1}||(T(t)-I)x||)^p \frac{dt}{t}\}^{\frac{1}{p}} \\
&= \frac{K}{1-2^{-\theta}}\{\int_0^\infty (t^{\theta-1}||(T(t)-I)x||)^p \frac{dt}{t}\}^{\frac{1}{p}}.
\end{aligned}$$

On the other hand, since

$$||(T(t)-I)x|| = ||\int_0^t AT(s)xds|| \leq \int_0^t ||AT(s)x||ds,$$

we have

$$\{\int_0^\infty (t^{\theta-1}||(T(t)-I)x||)^p \frac{dt}{t}\}^{\frac{1}{p}}$$
$$\leq \{\int_0^\infty (t^{\theta-1}\int_0^t ||AT(s)x||ds)^p \frac{dt}{t}\}^{\frac{1}{p}}.$$

from Lemma 3.1, it follows

$$\{\int_0^\infty (t^{\theta-1}||(T(t)-I)x||)^p \frac{dt}{t}\}^{\frac{1}{p}}$$
$$\leq \frac{1}{1-\theta}\{\int_0^\infty (t^{\theta}||AT(t)x||)^p \frac{dt}{t}\}^{\frac{1}{p}},$$

hence, the proof is complete.

COROLLARY 3.1. *Let $T(t)$ be an analytic semigroup with generator A in X. Then*

$$(D(A),X)_{\theta,\frac{1}{\theta}} = \{x \in X . \int_0^t ||AT(t)x||^{\frac{1}{\theta}}dt < \infty\}$$

In particular, if $\theta = \frac{1}{2}$ then

$$L^2(0,\infty;D(A)) \cap W^{1,2}(0,\infty,X) \subset C([0,\infty),(D(A),X)_{\frac{1}{2},2})$$

where

$$(D(A),X)_{2,\frac{1}{2}} = \{x \in X \quad \int_0^\infty ||AT(t)x||^2 < \infty\}$$

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