Honam Mathematical J. **20**(1998), No. 1,pp. 67–78

NONEXISTENCE OF GLOBAL SOLUTIONS FOR THE SEMILINEAR PARABOLIC SYSTEM

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1. Introduction

It is well-known that solutions of a semilinear parabolic system

$$\begin{aligned} &u_t - u_{xx} = f(v) \qquad (-a < x < a,\ t > 0),\\ &v_t - v_{xx} = g(u) \qquad (-a < x < a,\ t > 0) \text{ with}\\ &u(\pm a, t) = 0\ (t > 0), \quad u(x,0) = u_0(x) \quad (-a < x < a),\\ &v(\pm a, t) = 0\ (t > 0), \quad v(x,0) = v_0(x) \quad (-a < x < a) \end{aligned}$$

may blow up in finite time if the reaction terms f and g are positive, increasing and superlinear and if initial data u_0 and v_0 satisfy that

$$\begin{aligned} &u_0'(x) \le 0, \quad v_0'(x) \le 0, \quad \text{if } 0 < x < a,\\ &u_0(0) > 0, \quad v_0(0) > 0, \quad u_0(a) = 0 = v_0(a) \end{aligned}$$

(see Friedman and Giga[1] and the references therein).

More recently Gang and Sleeman [5] and Nam, Ju and Kim [6] established a result concerning the blow-up problem of the more general form than above system.

In this paper, we consider a semilinear parabolic system

$$(1) \qquad \begin{aligned} u_t - u_{xx} &= f(u,v) \qquad (-a \le x \le a,\ t \ge 0)\\ v_t - v_{xx} &= g(u,v) \qquad (-a \le x \le a,\ t \ge 0) \end{aligned}$$

Received May 1, 1998.
1991 AMS Subject Classification : 35K15, 35K20, 34D30.
Key words and phrases : life span, blows up, semilinear parabolic system.
This paper was supported by Mokpo National University Research Fund in 1997.

with boundary conditions

$$(2) \qquad u(\pm a,t)=0,\ v(\pm a,t)=0, \quad t\geq 0,$$

and initial conditions

$$(3) \qquad u(x,0)=u_0(x),\ v(x,0)=v_0(x), \quad -a\leq x\leq a.$$

It is well-known that classical solutions of the system (1) may blow up in finite time if f and g satisfy certain conditions (see Gang and Sleeman[5]). We define

$$T^* = \sup\{T>0 \,|\, (u,v) \text{ is bounded and solves (1) in } [-a,a]\times(0,T)\}.$$

Then T^* is called *the life span* of solutions (u,v). If T is infinite, the solutions are global. If T is finite one has

$$(4) \qquad \lim_{t\to T^*} ||u(x,t)||_\infty = \infty \text{ or } \lim_{t\to T^*} ||v(x,t)||_\infty = \infty,$$

since otherwise solutions could be extended beyond T. When (4) holds we say that the solution *blows up* in finite time.

Here we are interested in the question of global existence and nonexistence or life span of the solutions of (1).

In section 2 we shall establish a blow-up for solutions (u,v) of (1) satisfying (2) and (3). The main idea is based on the method of Gang and Sleeman[5].

In section 3 for given special reaction terms in (1) we obtain the life span such that solutions blow up in finite time.

2. Preliminaries

In this section we give a general theorem concerning the nonexistence of global solutions to the system (1) satisfying the initial-boundary conditions.

Suppose that $u(x,t)$ and $v(x,t)$ are local solutions to the system (1). From the standard theorem of PDE(see Smoller [7]), the existence of such solutions are guaranteed on $[-a,a]\times[0,T)$ for some $T>0$.

Let

$$
\begin{aligned}
g_1(t) &= \int_{-a}^{a} \phi(x)u(x,t)dx, \ 0 \le t < T, \\
g_2(t) &= \int_{-a}^{a} \phi(x)v(x,t)dx, \ 0 \le t < T
\end{aligned} \tag{5}
$$

where $\phi(x)$ is the first normalized eigenfunction which solves the eigenvalue problem

$$
\begin{aligned}
\phi_{xx} + r\phi = 0, \quad & x \in (-a,a) \\
\phi(\pm a) = 0. &
\end{aligned} \tag{6}
$$

Then

$$
\begin{aligned}
g_1'(t) &= \int_{-a}^{a} \phi(x)u_t(x,t)dx \\
&= \int_{-a}^{a} \phi(x)u_{xx}(x,t)dx + \int_{-a}^{a} \phi(x)f(u,v)dx.
\end{aligned}
$$

An integration by parts and (6) shows that

$$
g_1'(t) = -rg_1(t) + \int_{-a}^{a} \phi(x)f(u,v)dx. \tag{7}
$$

Similarly,

$$
g_2'(t) = -rg_2(t) + \int_{-a}^{a} \phi(x)g(u,v)dx. \tag{8}
$$

Suppose that there exists a function $G_i : \mathbb{R}^2 \to \mathbb{R}$ $(i = 1,2)$ such that the inequality

$$
\begin{aligned}
\int_{-a}^{a} \phi(x)f(u,v)dx &\ge G_1(g_1(t), g_2(t)), \\
\int_{-a}^{a} \phi(x)g(u,v)dx &\ge G_2(g_1(t), g_2(t))
\end{aligned} \tag{9}
$$

holds. If such a function G_i can be suitably determined, then from (7) and (8) we have the following differential inequalities

$$(10) \qquad \begin{aligned} g_1'(t) &\geq -rg_1(t) + G_1(g_1(t), g_2(t)), \\ g_2'(t) &\geq -rg_2(t) + G_2(g_1(t), g_2(t)) \end{aligned}$$

with initial conditions

$$(11) \qquad \begin{aligned} g_1(0) &= \int_{-a}^{a} \phi(x)u(x,0)dx, \\ g_2(0) &= \int_{-a}^{a} \phi(x)v(x,0)dx. \end{aligned}$$

Now we consider the system of the following ordinary differential equations associated with (10) and (11);

$$(12) \qquad \begin{aligned} y_1'(t) &= -ry_1(t) + G_1(y_1(t), y_2(t)), \\ y_2'(t) &= -ry_2(t) + G_2(y_1(t), y_2(t)) \end{aligned}$$

with initial conditions

$$(13) \qquad \begin{aligned} y_1(0) &= g_1(0), \\ y_2(0) &= g_2(0). \end{aligned}$$

Notice that blow-up behavior of solutions to the differential inequality (10) satisfying (11) associated with the system (12) is achived if the functions G_i can be chosen to be quasimonotone in the following definition.

DEFINITION 2.1. A function $G : D \subset \mathbb{R}^2 \to \mathbb{R}^2$ is said to be *quasimonotone nondecreasing* if $\frac{\partial G_i}{\partial y_j} \geq 0$ for $i \neq j$ where $G = (G_1, G_2), y = (y_1, y_2)$.

The next results is due to Gang and Sleeman[5].

LEMMA 2.2. *Suppose that $G(y) = (G_1(y_1, y_2), G_2(y_1, y_2))$ is quasimonotone nondecreasing and that solutions to the system (12) and (13) exist only in a finite time interval. That is, there exists a time $\tau > 0$ such that the solution to the system (12) exists for $t \in [0, \tau)$ and $y_1^2(t) + y_2^2(t) \to \infty$ as $t \to \tau$. Then the solution of (10) exists for only a finite time interval $0 \leq t < t_0 (t_0 \leq \tau)$ and $g_1^2(t) + g_2^2(t) \to \infty$ as $t \to t_0$.*

THEOREM 2.3. *Suppose that the inequalities (9) hold in which $G(y) = (G_1(y), G_2(y))$ is quasimonotone nondecreasing and that the solutions of (12) blow up in finite time. Then the solutions to the system (1) satisfying the initial-boundary coditions blow up in finite time.*

Proof. It follows from a direct consequence of Lemma 2.2.

An example which satisfies the inequalities (9) is as follows;

EXAMPLE 2.4. Let $f(u,v) = \lambda u + \lambda^2(u^2 - v^2)$ and $g(u,v) = \lambda v - 2\lambda^2 uv$ $(\lambda > 0)$ and let $\phi(x)$ be the first eigenfunction of the eigenvalue problem (6). By using Hölder's inequality, we obtain

$$\int_{-a}^{a} \phi f(u,v)dx \geq \lambda g_1(t) + \lambda^2 \left(\int_{-a}^{a} \phi u \, dx\right)^2 - \lambda^2 \int_{-a}^{a} \phi v^2 dx.$$

Let $R \subset \Omega_c = \{(u,v) \,|\, u \geq 0, 0 \leq v \leq c\}$ be an invariant region(See Lemma 3.2). Then

$$\int_{-a}^{a} \phi v^2 dx \leq c^2.$$

Hence,

$$\int_{-a}^{a} \phi f(u,v)dx \geq \lambda g_1(t) + \lambda^2 g_1^2(t) - \lambda^2 c^2$$

for all u, v in R.

Similarly

$$\int_{-a}^{a} \phi g(u,v)dx \geq \lambda g_2(t) - 2\lambda^2 c g_1(t)$$

for all u, v in R.

If we choose $G_1(y_1, y_2) = \lambda y_1 + \lambda^2(y_1^2 - c^2)$ and $G_2(y_1, y_2) = \lambda y_2 - 2\lambda^2 c y_1$, then we see that G_1 is quasimonotone nondecreasing since $\frac{\partial G_1}{\partial y_2} = 0$. Consequently it is sufficient to consider the associated ordinary differential inequality;

$$g_1'(t) \geq -r g_1(t) + \lambda g_1(t) + \lambda^2(g_1^2(t) - c^2)$$
$$g_1(0) = \int_{-a}^{a} \phi(x) u_0(x) dx.$$

3. Blow-up of the Solution for the Special Reaction Terms

In this section we analyse the following semilinear parabolic system

$$(\mathrm{P}) \qquad \begin{aligned} u_t - u_{xx} &= \lambda u + \lambda^2(u^2 - v^2), \qquad (-a \le x \le a,\ t \ge 0) \\ v_t - v_{xx} &= \lambda v - 2\lambda^2 uv, \qquad (-a \le x \le a,\ t \ge 0) \end{aligned}$$

satisfying (2) and (3), where λ is a positive constant.

First, we deduce some geometrical and qualitative properties of (P).

LEMMA 3.1. *The system (P) is symmetric about $u-$axis and invariant under a rotation through an angle of $\frac{2\pi}{3}$.*

Proof. We introduce the following notations;

$$\begin{aligned} J &= \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}, \\ A &= \begin{pmatrix} u \\ v \end{pmatrix}, A^t = (u, v), \\ F(A^t) &= \begin{pmatrix} f(A^t) \\ g(A^t) \end{pmatrix} = \begin{pmatrix} \lambda u + \lambda^2(u^2 - v^2) \\ \lambda v - 2\lambda^2 uv \end{pmatrix}. \end{aligned}$$

Then we can write (P) in the form

$$JA = F(A^t).$$

Let R_1 be a reflection about the $u-$axis and let R_2 be a rotation through an angle θ, i.e.,

$$R_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, R_2 = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

Then we can prove the followings;

$$J(R_1 A) = F\left((R_1 A)^t\right)$$

and

$$J\left(R_2(\frac{2\pi}{3})A\right) = F\left[\left(R_2(\frac{2\pi}{3})A\right)^t\right].$$

This complete the proof of our lemma.

By Lemma 3.1, we will only consider the system (P) in region Γ which is bounded from below by $v = 0$ and from the left by $v = \sqrt{3}u$ rather than the (u, v)−plane.

LEMMA 3.2. *Let*

$$\Gamma_c = \{\,(u,v) \mid u \geq 0, 0 \leq v \leq c, v \leq \sqrt{3}u, c \geq \frac{\sqrt{3}}{2\lambda}\,\}.$$

If any solution(u, v) of (P) satisfies all of its boundary and initial values in Γ_c, then $(u, v) \in \Gamma_c$ for all (x, t) at which the solution is defined. That is, Γ_c is an invariant region for the system (P).

Proof. The rescaling functions

$$w = \sqrt{3}u - v \text{ and } z = v \tag{14}$$

satisfy the system

$$\begin{aligned} w_t - w_{xx} &= \frac{\lambda}{\sqrt{3}} w(\sqrt{3} + \lambda w + 4\lambda z) \\ z_t - z_{xx} &= \frac{\lambda}{\sqrt{3}} z(\sqrt{3} - 2\lambda w - 2\lambda z). \end{aligned} \tag{15}$$

The rescaling function (14) transform the region Γ in (u, v)−plane into the whole of the first quadrant in the (w, z)−plane and the region Γ_c becomes a strip

$$\Omega_c = \{\,(w,z) \mid w \geq 0, 0 \leq z \leq c, c \geq \frac{\sqrt{3}}{2\lambda}\,\}$$

in the first quadrant of the (w, z)−plane.

Notice that from the local existence theorem (see [7]), it is well-known fact that there exists a solution to the system (P) for any initial value $(w_0, z_0) \in \Omega_c$ and for $(x,t) \in [-a,a] \times [0,\epsilon)$ where $\epsilon > 0$ is sufficiently small. Also, we may assume that $[0,\epsilon)$ is the maximal time interval of the existence of the solution for (P).

Using Theorem 14.11 in [7] we know that the solution can not escape the region Ω_c by crossing the lower boundary $z = 0$ and the left boundary $w = 0$. Therefore we must only show that the solution can not leave by crossing the upper boundary $z = c$.

To show this, setting $I\left(z(x,t)\right) = z(x,t) - c(c \geq \frac{\sqrt{3}}{2\lambda})$, we obtain

$$\Omega_c = \{\, (w,z) \mid w \geq 0, z \geq 0, I \leq 0 \,\}.$$

Assume that there is an $x_0 \in (-a,a)$ and $t_0 \in (0,\epsilon)$ such that

$$\begin{aligned} &I\left(z(x,t)\right) < 0, \quad (x,t) \in (-a,a) \times (0,t_0) \\ &I\left(z(x_0,t_0)\right) = 0. \end{aligned}$$

Then since $\frac{\partial}{\partial t} I\left(z(x,t)\right) = z_{xx} + \frac{\lambda}{\sqrt{3}} z(\sqrt{3} - 2\lambda w - 2\lambda z)$ from (15), (16)

$$\begin{aligned} \frac{\partial}{\partial t} I\left(z(x_0,t_0)\right) &= z_{xx}(x_0,t_0) - \frac{2}{\sqrt{3}}\lambda^2 wz - \frac{2}{\sqrt{3}}\lambda^2 z(z - \frac{\sqrt{3}}{2\lambda}) \\ &= z_{xx}(x_0,t_0) - \frac{2}{\sqrt{3}}\lambda^2 wz - \frac{2}{\sqrt{3}}\lambda^2 \delta(\frac{\sqrt{3}}{2\lambda} + \delta) \\ &< z_{xx}(x_0,t_0) \end{aligned}$$

where $\delta = z(x_0,t_0) - \frac{\sqrt{3}}{2\lambda} \geq 0$.

Let $K(x) = z(x,t_0) - c$. Then $K(x_0) = I(z(x_0,t_0)) = 0$ and $K'(x) = z_x(x,t_0)$.

If $K'(x_0) = z_x(x_0,t_0)$ is positive in some interval $(x_0, x_0 + \eta)$, then $K(x)$ is also positive in that interval $(x_0, x_0 + \eta)$ for η sufficiently small.

Therefore,

$$z(x,t) - c > 0 \text{ for } (x,t) \in (x_0, x_0 + \eta) \times (t_0, t_0 + \eta).$$

This is a contradiction to $z(x,t) - c < 0$ for $(x,t) \in (-a,a) \times (0,t_0)$ So $K'(x_0) > 0$ is impossible.

Similarly, we can show that $K'(x_0) < 0$ is impossible. Hence $K'(x_0) = 0$.

Notice that since $K(x_0) = 0$, $K'(x_0) = 0$, and $K(x) \leq 0$, $K(x)$ has a local maximum at $x = x_0$ and $K''(x_0) = z_{xx}(x_0, t_0) \leq 0$. So, from the inequality (16),

$$\frac{\partial}{\partial t} I(z(x_0, t_0)) < z_{xx}(x_0, t_0) \leq 0$$

which means that the solution can not leave Ω_c by crossing the upper boundary $z = c$. Therefore Ω_c is invariant.

Here, we establish the nonexistence of the global solution for the system (P).

First, we consider the system (P) with initial condition

$$(17) \qquad (u_0(x), v_0(x)) \in \Gamma_c \text{ for } x \in [-a, a], c \geq \frac{\sqrt{3}}{2\lambda}.$$

From the local existence theorem for parabolic system, we see that there is a solution to the initial value problem (P) satisfying (17) which exists for $0 \leq t < T$. From Lemma 3.2 and the initial condition (17), we obtain that

$$(18) \qquad u(x,t) \geq 0, 0 \leq v(x,t) \leq c, c \geq \frac{\sqrt{3}}{2\lambda}$$

for $(x, t) \in [-a, a] \times [0, T)$.

LEMMA 3.3. *For the eigenvalue problem (6), $r = \frac{\pi^2}{16a^2}$ is the least eigenvalue and $\phi(x) = \frac{\pi}{4\sqrt{2}a}(\cos\frac{\pi}{4a}x + \sin\frac{\pi}{4a}x)$ is the corresponding normalized eigenfunction.*

From Example 2.4, we consider the following system of ordinary differential inequalities

$$(19) \qquad \begin{aligned} y_1'(t) &\geq -ry_1(t) + \lambda y_1(t) + \lambda^2(y_1^2(t) - c^2), \\ y_2(t) &\geq -ry_2(t) + \lambda y_2(t) - 2\lambda^2 c y_1(t) \end{aligned}$$

with initial conditions

$$
\begin{aligned}
y_1(0) &= \int_{-a}^{a} \phi(x)u_0(x)dx, \\
y_2(0) &= \int_{-a}^{a} \phi(x)v_0(x)dx.
\end{aligned} \tag{20}
$$

THEOREM 3.4. *If the system of ordinary differential inequalities (19) satisfying (20) has a unbounded solution, then the system (P) satisfying (17) blows up in finite time.*

Proof. Since the first inequality of (19) does not depend on the variable y_2, we need only consider the following equation associated with the first inequality;

$$
\begin{aligned}
z' &= (\lambda - r)z + \lambda^2(z^2 - c^2) \\
z(0) &= y_1(0).
\end{aligned} \tag{21}
$$

Since $(\lambda - r)^2 + 4\lambda^4 c^2$ is positive, we can separate the right-hand side of the first equation of (21) into

$$\lambda^2 z^2 + (\lambda - r)z - \lambda^2 c^2 = \lambda^2 (z - z_1)(z - z_2),$$

where $z_1 + z_2 = \frac{r-\lambda}{\lambda^2}$, $z_1 z_2 = -c^2$, and $z_1 < 0 < z_2$.

Consequently, (21) implies that

$$z' = \lambda^2 (z - z_1)(z - z_2) \tag{22}$$

and also z_1 and z_2 are the equilibria of (22) in which $z = z_1$ is stable while $z = z_2$ is unstable.

In the region below $z = z_2$, all the solutions of (21) exist globally. So we will only consider the region above $z = z_2$, i.e., $z > z_2 > 0 > z_1$.

Integrating (22) from 0 to t in this region, we obtain

$$\frac{1}{z_2 - z_1}\left(\frac{dz}{z - z_2} - \frac{dz}{z - z_1}\right) = \lambda^2 dt$$

or

$$\frac{1}{z_2 - z_1}\int_{z_0}^{z}\frac{dz}{z-z_2} - \frac{1}{z_2 - z_1}\int_{z_0}^{z}\frac{dz}{z-z_1} = \int_0^t \lambda^2 dt$$

where $z_2 < z_0 < z$. So

$$\frac{z-z_2}{z-z_1} = \left(\frac{z_0 - z_2}{z_0 - z_1}\right) e^{\lambda^2(z_2-z_1)t}$$

or

$$z = \frac{z_2(z_0 - z_1) - z_1(z_0 - z_2)e^{\lambda^2(z_2-z_1)t}}{(z_0 - z_1) - (z_0 - z_2)e^{\lambda^2(z_2-z_1)t}}$$

where $z_2(z_0 - z_1) - z_1(z_0 - z_2)e^{\lambda^2(z_2-z_1)t}$ is positive.

On solving the equation $(z_0 - z_1) - (z_0 - z_2)e^{\lambda^2(z_2-z_1)t} = 0$, we obtain the life span $t = T^*$ of the system (P),

$$T^* \equiv T(z_0, \lambda) = \frac{1}{\lambda^2(z_2 - z_1)}\left[ln(z_0 - z_1) - ln(z_0 - z_2)\right].$$

Since $\frac{1}{z_2 - z_1} > 0$ and $\frac{z_0 - z_1}{z_0 - z_2} > 1$, T is positive. Hence we have proved.

LEMMA 3.5. *If $z_0 > z_2$, then the solution of (21) exists only in a finite time interval $0 \le t < T(z_0, \lambda)$ with*

$$\lim_{t \to T} z(t) = \infty.$$

REMARK. In order to obtain an explicit condition for the blow-up of the solution for the system (P) satisfying (17), we need to express z_0 and z_2 in terms of λ, r, ϕ, and (u_0, v_0). First,

$$z_0 = y_1(0) = \int_{-a}^{a} \phi(x)u_0(x)dx.$$

From $\lambda^2 z^2 + (\lambda - r)z - \lambda^2 c^2 = 0$, we have the larger root

$$z_2 = \frac{r - \lambda + \sqrt{(\lambda - r)^2 + 4\lambda^4 c^2}}{2\lambda^2}$$

where $r = \frac{\pi^2}{16a^2}$, $c = \frac{\sqrt{3}}{2\lambda} + \delta$, $\delta \ge 0$. So,

(23)

$$z_2 = \frac{\pi^2 - 16\lambda a^2}{32a^2\lambda^2} + \frac{1}{2\lambda^2}\left[(\frac{\pi^2 - 16\lambda a^2}{16a^2})^2 + 4\lambda^4(\frac{\sqrt{3}}{2\lambda} + \delta)^2\right]^{\frac{1}{2}}.$$

THEOREM 3.6. *If the blow-up condition*

$$\int_{-a}^{a}\phi(x)u_0(x)dx > \frac{\pi^2-16\lambda a^2}{32a^2\lambda^2}+\frac{1}{2\lambda^2}\left[(\frac{\pi^2-16\lambda a^2}{16a^2})^2+4\lambda^4(\frac{\sqrt{3}}{2\lambda}+\delta)^2\right]^{\frac{1}{2}}$$

holds where

$$\phi(x)=\frac{\pi}{4\sqrt{2}a}(cos\frac{\pi}{4a}x+sin\frac{\pi}{4a}x)$$

and $\delta > 0$ is an arbitrary constant, then the solution of (P) satisfying (17) blows up in finite time.

REMARK. If $a=\frac{\pi}{4}$, then $z_0 > z_2$ implies that

$$\frac{1}{\sqrt{2}}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}(cosx+sinx)dx > \frac{1-\lambda+\sqrt{(1-\lambda)^2+4\lambda^4c^2}}{2\lambda^2}.$$

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