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RELATIONS BETWEEN BANACH FUNCTION ALGEBRAS AND FRÉCHET FUNCTION ALGEBRAS

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Abstract In this paper we define the concept of Fréchet function algebras on hemicompact spaces. So we show that under certain condition they can be represented as a projective limit of Banach function algebras. Then the class of Fréchet Lipschitz algebras on hemicompact metric spaces are defined and their relations with the class of lipschitz algebras on compact metric spaces are studied.

0. Introduction

A Fréchet algebra is an LMC-algebra which is moreover complete and metrizable. So its topology can be defined by a sequence (p_n) of submultiplicative seminorms. Without loss of generality we can assume that $p_n \leq p_{n+1}$. We denote a Fréchet algebra A with this sequence of seminorms by $(A, (p_n))$. The *spectrum* of the Fréchet algebra $(A, (p_n))$, denoted by M_A, is the space of all non-zero, continuous complex homomorphisms on A. Also $\widehat{A}$ is the set of all Gelfand transforms $\hat{f}$ of $f \in A$. Let A_n be the completion of $A/\ker\ p_n$ with respect to the norm $p_n(f + \ker\ p_n) = p_n(f)$. Then A is the projective limit of the sequence (A_n) of Banach algebras ($A =$ projlim A_n), and M_A can be identified in a natural way with $\bigcup M_{A_n}$ [5].

A Hausdorff topological space X is called hemicompact if there exists an increasing sequence (K_n) of compact subsets of X such

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that each compact subset K of X is contained in K_n, for some n. We call such sequence (K_n) an *admissible exhaustion* of X. It is shown in [5] that M_A is a hemicompact space with (M_{A_n}) as an admissible exhaustion.

For the notations, definitions and known results one can refer to [3].

1. Main results

DEFINITION 1.1. Let X be a hemicompact space. A subalgebra A of $C(X)$ which contains the constants and separates the points of X is called a *Fréchet function algebra* on X, if it is a Fréchet algebra with respect to some topology such that for each $x \in X$, the evaluation homomorhism φ_x is continuous. If the topology of A is the compact-open topology , then A is called a *uniform Fréchet algebra* on X (in this case $\varphi_x \in M_A$ automatically).

Clearly every Fréchet function algebra is semisimple. Conversely each commutative semisimple Fréchet algebra is a Fréchet function algebra on its spectrum.

REMARK. The definition of Fréchet function algebras and uniform Fréchet algebras which are used in this paper, is slightly different from the definition in [2]. In [2] when A is a Fréchet function algebra on X, the Gelfand topology coincide on the topology of X. But with the definition of uniform Fréchet algebra in [3] this is not true [3, page 98]. We follow the same definition for uniform Fréchet algebras as in Goldmann's book and define the concept of Fréchet function algebras in a similar way.

LEMMA 1.2. *Let $(A,(p_n))$ be a Fréchet function algebra on a hemicompact space X, and let (K_n) be an admissible exhaustion of X. Then the identity map i from the Fréchet algebra $(A,(p_n))$ into the (metrizable) LMC-algebra $(C(X),(\|.\|_{K_n}))$ is continuous.*

Proof. Since $X \longrightarrow M_A; x \longmapsto \varphi_x$ is continuous, the subset $\{\varphi_x : x \in K_n\}$ of M_A is compact, for each n. Hemicompactness of M_A implies that there exists some integer m such that $\{\varphi_x : x \in K_n\} \subseteq M_{A_m}$. It is well known that M_{A_m} can be identified with

$\{\varphi \in M_A : |\varphi(f)| \leq p_m(f), f \in A\}$. So for each n there exists some m such that

$$\|f\|_{K_n} \leq \|\hat{f}\|_{M_{A_m}} \leq p_m(f) \tag{1}$$

for all $f \in A$, and this implies the continuity of i.

If X is a hemicompact k-space, then $(C(X), (\|.\|_{K_n}))$ is a uniform Fréchet algebra on X. So the closure of a Fréchet function algebra $(A, (p_n))$ in $C(X)$, $\bar{A}$, is a uniform Fréchet algebra on X. Since $i : (A, (p_n)) \longrightarrow (\bar{A}, (\|.\|_{K_n}))$ is a continuous monomorphism with a dense range, its transpose map $i^* : M_{\bar{A}} \longrightarrow M_A; \psi \longmapsto \psi|_A$ is continuous and injective. In general it is not true that i^* is surjective. The following theorem, which is a generalization of [4], gives a condition for surjectivity of i^*.

We recall that a map between two topological spaces is called *proper* if the inverse image of each compact subset is compact.

THEOREM 1.3. *Let $(A, (p_n))$ and $(B, (q_n))$ be Fréchet function algebras on X and Y respectively, and let $T : A \longrightarrow B$ be a continuous monomorphism with a dense range. Then the continuous and injective map $T^* : M_B \longrightarrow M_A; \psi \longmapsto \psi \circ T$ is proper and surjective if and only if for each m there exists some n such that*

$$\|\hat{f}\|_{M_{A_m}} \leq q_n(T(f)) \tag{2}$$

for all $f \in A$.

Proof. Suppose that inequality (2) holds, and let $\varphi \in M_A$. For each $g \in B$, there exists a sequence (f_n) in A such that $T(f_n) \longrightarrow g$ in $(B, (q_n))$. Choose m such that $\varphi \in M_{A_m}$. By the hypothesis there exists some n such that

$$|\varphi(f_i) - \varphi(f_j)| \leq \|\hat{f}_i - \hat{f}_j\|_{M_{A_m}} \leq q_n(T(f_i) - T(f_j))$$

for every i and j. This shows that $(\varphi(f_i))$ is a Cauchy sequence, and therefore we can define ψ on B by $\psi(g) = \lim_{i\to\infty}\varphi(f_i)$. It is easy to see that ψ is well defined and it is a complex homomorphism with $\psi \circ T = \varphi$.

Since $T(A)$ is dense in B, the definition of ψ implies that $|\psi(g)| \leq q_n(g)$, for all $g \in B$. So $\psi \in M_B$.

The above argument also shows that

$$(3) \quad T^{*^{-1}}(M_{A_m}) \subseteq M_{B_n} = \{\psi \in M_B : |\psi(g)| \leq q_n(g), g \in B\}.$$

So if $K \subseteq M_A$ is compact then the closed subset $T^{*^{-1}}(K)$ of M_B is contained in M_{B_n} for some n. Hence it is a compact subset of M_B and T^* is a proper map.

Conversely suppose that T^* is a surjective and proper map. Then for each m, $T^{*^{-1}}(M_{A_m})$ is compact and so there exists n with $T^{*^{-1}}(M_{A_m}) \subseteq M_{B_n}$. This implies the desired inequality.

REMARKS. (i) When (2) holds we say that $M_A = M_B$ as sets. Since in this case T^* is a proper map, (3) implies that the restriction of $T^{*^{-1}}$ to each M_{A_m} is continuous. In other words $T^{*^{-1}}$ is sequentially continuous. So if (2) holds and M_A is a k-space, then T^* is a homeomorphism.

(ii) In Theorem 1.3, if A and B are Banach function algebras on M_A and M_B respectively, with A regular, then T^* is always surjective [6, Theorem 3.7.5]. This is not true in Fréchet algebra case, even if A is a regular Banach function algebra on M_A. For example let X be a hemicompact, noncompact space and let A be the Banach algebra (Lip $(X, \alpha), \|.\|_\alpha)$, and let B be the Fréchet algebra (FLip $(X, \alpha), (p_K))$ which will be defined in Section 2. Then the (continuous) identity map $i : A \longrightarrow B$ has a dense range and $M_B \cong X$ (Theorem 2.1), while X is dense in M_A [8].

Let $(A, (p_n))$ be a Fréchet algebra. Then A is semisimple if each A_n is semisimple. The converse statement is not true [7]. But it is well known that each uniform Fréchet algebra is the projective limit of a sequence of uniform (Banach) algebras [3]. In the following we obtain a similar result for Fréchet function algebras with additional assumptions.

Let $i(n) \geq n$ be the smallest integer such that

$$\|f\|_{K_n} \leq \|\hat{f}\|_{M_{A_{i(n)}}} \leq p_{i(n)}(f)$$

holds for all $f \in A$ (Lemma 1.3). Then we have the following theorem :

THEOREM 1.4. *Let* (K_n) *be an admissible exhaustion of* X. *Then for each Fréchet function algebra* $(A, (p_n))$ *on* X *there exists a sequence* (A_{K_n}) *of Banach algebras in which* A_{K_n} *contains* $A|_{K_n} \subseteq C(K_n)$ *as a dense subalgebra and* A *is dense in* projlim A_{K_n}. *If* (p_n) *is such that for each* $f \in A$, $f|_{K_n} = 0$ *implies* $p_{i(n)}(f) = 0$ *then* $A =$ projlim A_{K_n} *(topologically and algebraically).*

Proof. Define p'_n on $A|_{K_n}$ by

$$p'_n(f|_{K_n}) = \inf\{p_{i(n)}(g) : g|_{K_n} = f|_{K_n}, g \in A\}.$$

If $p'_n(f|_{K_n}) = 0$, then for each $\epsilon > 0$ there exists $g \in A$ with $g|_{K_n} = f|_{K_n}$ and $p_{i(n)}(g) < \epsilon$. So $\|f\|_{K_n} = \|g\|_{K_n} \leq p_{i(n)}(g) < \epsilon$ and hence $f|_{K_n} = 0$. This shows that p'_n is a norm on $A|_{K_n}$.

Let A_{K_n} be the completion of $A|_{K_n}$ with respect to p'_n. Then A_{K_n} is a Banach algebra and $A|_{K_n}$ is dense in A_{K_n}, and we have

$$p'_n(f|_{K_n}) \leq p'_{n+1}(f|_{K_{n+1}}) \qquad (f \in A).$$

It is easy to see that p'_n can be extended to A_{K_n}, while preserving the same properties as on $A|_{K_n}$. Let $\pi_n : A \longrightarrow A_{K_n}; f \longmapsto f|_{K_n}$, and for $n \geq m$, let $\pi_{n,m} : A_{K_n} \longrightarrow A_{K_m}$ be the extension of $f|_{K_n} \longmapsto f|_{K_m}$ to A_{K_n}. Consider $\psi_n = \pi_{n+1,n}$, then (A_{K_n}, ψ_n) is a dense projective system of Banach algebras. So projlim A_{K_n} is a Fréchet algebra with the family of seminorms

$$p^*_k((f_n)_n) = p'_k(f_k) \qquad (k \in \mathbb{N}).$$

Define $T : A \longrightarrow$ projlim A_{K_n} by $T(f) = (f|_{K_n})_n$. For each $f \in A$, we have

$$p^*_l(T(f)) = p'_l(f|_{K_l}) \leq p_{i(l)}(f) \qquad (l \in \mathbb{N}).$$

So T is continuous. Clearly T is injective, and it is not difficult to see that T has a dense range.

For the second part, we follow the method which is used for uniform Fréchet algebras. Let $(f_n)_n \in$ projlim A_{K_n}. For each integer k there exists $g_k \in A$ with $p'_k(f_k - \pi_k(g_k)) < 1/k$. So for $k \geq n$

$$p'_n(f_n - \pi_n(g_k)) = p'_n(\pi_{k,n}(f_k) - \pi_{k,n}(\pi_k(g_k)))$$
$$\leq p'_k(f_k - \pi_k(g_k)) < 1/k.$$

Fix $m \in \mathbb{N}$ and let $k \geq l \geq m$. By the assumption $p'_l(f|_{K_l}) = p_{i(l)}(f)$, for each $f \in A$. As in the proof of Theorem 3.3.7 in [3] we can conclude that (g_n) is a Cauchy sequence in A and for each k, $p^*_k(T(g) - (f_n)_n) = 0$, in which $g = \lim g_n$. Therefore $T(g) = (f_n)_n$. This shows that T is surjective and hence a homeomorphism, by the open mapping theorem.

Remark. In the above theorem if for each n, $(A|_{K_n}, p'_n)$ is complete or if its completion A_{K_n} can be considered as a subalgebra of $C(K_n)$, then (A_{K_n}) is a sequence of Banach function algebras. Examples of these situations are given in the next section.

2. Examples

Let (X, d) be a metric space, and let $\alpha \in (0, 1]$. Consider the Banach algebra

$$\text{Lip }(X, \alpha) = \{f \in C_b(X) : \sup_{x \neq y} \frac{|f(x) - f(y)|}{d^\alpha(x, y)} < \infty\}$$

under the norm

$$\|f\|_\alpha = \|f\|_X + p_\alpha(f) \quad (f \in \text{ Lip }(X, \alpha)),$$

where $p_\alpha(f) = \sup_{x \neq y} \frac{|f(x)-f(y)|}{d^\alpha(x,y)}$ and $\|f\|_X = \sup_{x \in X} |f(x)|$.

It is well known that for compact metric space X, Lip (X, α) is a natural, regular, Banach function algebra on X. For $\alpha \in (0, 1)$

$$\text{lip }(X, \alpha) = \{f \in C_b(X) : \frac{|f(x) - f(y)|}{d^\alpha(x, y)} \to 0, \text{ as } d(x, y) \to 0\}$$

is a closed subalgebra of Lip (X, α), which is also a natural, regular Banach function algebra on X when X is compact [8].

Now consider an arbitrary metric space (X, d), and for each compact subset K of X define

$$(4) \qquad p_K(f) = \|f\|_K + p_\alpha(f|_K)$$

for all $f \in C(X)$ for which $p_\alpha(f|_K) < \infty$. Let τ be the topology defined on Lip (X, α) by the family (p_K) of seminorms. Clearly (Lip $(X, \alpha), \tau)$ is an LMC-algebra which is metrizable when X is a hemicompact space. But if X is a hemicompact and noncompact space, then by using the Carpenter's theorem (uniquness of topology for commutative semisimple Fréchet algebras) and the regularity of (Lip $(X, \alpha), \|.\|_\alpha)$ on its spectrum [8], one can prove that Lip (X, α) is not complete with respect to τ. Take FLip (X, α) to be the completion of Lip (X, α) with respect to τ. The following theorem characterizes the elements of FLip (X, α).

THEOREM 2.1. *Let (X, d) be an arbitrary metric space. Consider*

$$A = \{f \in C(X) : f|_K \in \text{ Lip } (K, \alpha), K \subseteq X \text{ is compact } \}.$$

with the topology defined by the family of seminorms (p_K) as in (4). Then

(i) Lip (X, α) is dense in A with respect to this topology.

(ii) $(A, (p_K))$ is a Fréchet algebra if and only if X is hemicompact.

(iii) If X is a hemicompact space with admissible exhaustion (K_n), then $A =$ projlim Lip (K_n, α) and $M_A \cong X$.

(iv) $(A, (p_K))$ is a Banach algebra if and only if X is compact.

Proof. (i) This is a consequence of the definition of the topology and proposition 1.4 in [8].

(ii) First assume that X is hemicompact. Let (f_n) be a Cauchy sequence in A, i.e. $f_n|_{K_m}$ is a Cauchy sequence in Lip (K_m, α), for each m. Let $f_n|_{K_m} \to \tilde{f_m}$ in Lip (K_m, α), and define f on X by $f(x) = \tilde{f_m}(x)$, whenever $x \in K_m$. Since X is a k-space, $f \in C(X)$. So $f \in A$ and $f_n \to f$ in A.

Conversely suppose that A is a Fréchet algebra. Since it is metrizable there exist a sequence (K_n) of compact subsets and a sequence of positive numbers (c_n), such that $U_n = \{f \in A : \|f\|_{K_n} + p_\alpha(f|_{K_n}) < c_n\}$ is a local base. We show that (K_n) is an admissible exhaustion of X. Let $K \subseteq X$ be compact. Then $V = \{f \in A : \|f\|_K + p_\alpha(f|_K) < 1\}$ is an open set. So there exists some n with $U_n \subseteq V$. If $x \in K \backslash K_n$ then $f(t) = d(t, K_n)/d(x, K_n)$ is an element of A, so that $f \in U_n$ but $f \notin V$ and this is a contradiction. Without loss of generality we can assume that (K_n) is an increasing sequence.

(iii) By proposition 1.4 in [8] we have $A|_{K_n} =$ Lip (K_n, α). Since for each n, $f|_{K_n} = 0$ implies $p_{K_n}(f) = 0$, Theorem 1.4 shows that $A =$ projlim Lip (K_n, α). So $M_A = \bigcup M_{\text{Lip }(K_n,\alpha)} = \cup K_n = X$ (as sets).

For each $x_0 \in X$, and $f_1, f_2, \cdots, f_n \in A$, it is clear that $V = \{x \in X : |f_i(x) - f_i(x_0)| < \epsilon, 1 \leq i \leq n\}$ is an open set in the metric topology. Converesely if $S \subseteq X$ is d–closed, so that $p \notin S$, then $f(x) = 1 - d(x, S)/d(p, S)$ is in A and $f(p) = 0$. Since $f|_S = 1$, $\{y \in X : |f(y) - f(p)| < 1/2\}$ is a neighbourhood of p in the Gelfand topology, which dose not intesect S. So S is closed in the Gelfand topology.

(iv) Suppose that A is a Banach algebra. Then clearly M_A is compact. (ii) implies that X is hemicompact, and by (iii) $M_A \cong X$. So X is compact.

The converse is immediate.

REMARKS. (i) In the sequel we assume that X is a hemicompact space and (K_n) is an admissible exhaustion of X. We also use p_n for p_{K_n}. Theorem 2.1 shows that

$$\text{FLip }(X, \alpha) = \{f \in C(X) : f|_{K_n} \in \text{ Lip }(K_n, \alpha), n \in \mathbb{N}\}.$$

Here it is not necessary to assume that $f \in C(X)$. Because the second condition implies that the restriction of f to each compact subset is continuous and since X is a k-space, f is continuous on X.

(ii) By using the fact that FLip $(X, \alpha)|_{K_n} =$ Lip (K_n, α) and Lip (K_n, α) is dense in $C(K_n)$, we can conclude that FLip (X, α) is dense in $C(X)$ with respect to the compact-open topology.

DEFINITION 2.2. For α, $0 < \alpha < 1$, consider the subalgebra

$$\text{Flip } (X, \alpha) = \{f \in C(X) : f|_{K_n} \in \text{ lip } (K_n, \alpha)\}$$

of FLip (X, α). It is easy to see that Flip (X, α) is a closed subalgebra of FLip (X, α), which is also a Fréchet function algebra on X.

THEOREM 2.3. *For each α, $0 < \alpha < 1$, Lip $(X, 1)$ (and so FLip $(X, 1)$) is dense in Flip (X, α).*

Proof. Clearly Lip $(X, 1) \subseteq$ Flip (X, α). Let $f \in$ Flip (X, α) and let $V = \{g : p_n(g - f) < 1/n\}$ be a neighbourhood of f in Flip (X, α). Then $f|_{K_n} \in$ lip $(K_n, \alpha) = \overline{\text{Lip } (K_n, 1)}$ [1]. So there exists $g \in$ Lip $(K_n, 1)$ with $\|g - f|_{K_n}\|_{K_n} + p_\alpha(g - f|_{K_n}) < 1/n$. Now g can be extended to a $\tilde{g} \in$ Lip $(X, 1) \subseteq$ FLip $(X, 1)$. Therefore $\tilde{g} \in V$.

COROLLARY 2.4. *$M_{\text{Flip } (X,\alpha)}$ is homeomorphic with X.*

Proof. Since $M_{\text{FLip } (X,1)}$ is homeomorphic with X, it is a k-space. A straightforward calculation shows that inequality (2) of Theorem 1.3 holds for $m = n$, and hence $M_{\text{Flip } (X,\alpha)} \cong X$.

THEOREM 2.5. *Let ψ_n : lip $(K_{n+1}, \alpha) \to$ lip (K_n, α) be the restriction map. Then (lip $(K_n, \alpha), \psi_n)$ is a dense projective system of Banach function algebras, and* Flip $(X, \alpha) =$ projlim lip (K_n, α).

Proof. By Theorem 1.4 it is enough to show that the completion of Flip $(X, \alpha)|_{K_n}$ with respect to p_n is lip (K_n, α). Let $f \in$ lip (K_n, α) and let $V = \{g \in \text{ lip } (K_n, \alpha) : \|g - f\|_{K_n} + p_\alpha(g - f) < 1/n\}$ be a neighbourhood of f. Then there exists $g \in$ Lip $(K_n, 1) \cap V$. Let $\tilde{g} \in$ Lip $(X, 1)$ be an extension of g on X. So $\tilde{g} \in$ Flip (X, α) and $\tilde{g}|_{K_n} \in V$. This shows that Flip $(X, \alpha)|_{K_n}$ is dense in lip (K_n, α).

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References

1. W.G. Bade, P.C. Curtis, JR, and H.G. Dales, *Amenability and weak amenability for Beurling and Lipschitz algebras*, Proc. London Math. Soc. **Vol. 55, No.3** (1987), 359-377.
2. H.G. Dales, *Banach algebras and automatic continuity, Oxford University Press, to appear.*
3. H. Goldmann, *Uniform Fréchet Algebras*, North-Holland, Amesterdam, 1990.
4. T.G. Honary, *Relations between Banach function algebras and their uniform closures*, Proc. Amer. Math. Soc. **109** (1990), 337-342.
5. E.A. Michael, *Locally m-convex topological algebras*, Mem. Amer. Math. Soc. **11** (1952).
6. C.E. Rickart, *General Theory of Banach Algebras*, Kieger publishing company, Huntington, 1960.
7. S. Rolewicz, *Example of semisimple m-convex B_0−algebra, which is not a projective limit of semisimple B−algebras*, Bull. Acad. Polon. Soc. **11** (1963), 459-462.
8. D.R. Sherbert, *The structure of ideals and point derivations in Banach algebras of Lipschitz functions*, Trans. Amer. Math. Soc. **111** (1964), 240-272.