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FEYNMAN INTEGRALS IN WHITE NOISE ANALYSIS

SOON-JA KANG

Dept. of Mathematics Education,
Chonnam National University, Kwangju 500-757, Korea.
E-mail: kangsj@chonnam.chonnam.ac.kr.

Abstract We first obtain the white noise calculus to the computation of Feynman integral for a generalized function, according to the definition of Feynman integrals by T. Hida and L. Streit. We next give the translation theorem for Feynman integral of a generalized function.

1. Introduction

In [7], Kallianpur, Kannan and Kannadikar defined sequential Feynman integrals on an abstract Wiener and Hilbert spaces and established the existence of both of analytic and sequential Feynman integrals for integrands belonging to some larger classes than Fresnel classes.

In [12], Yan and Luo studied the complex scaling transform in Wiener space (which is an analytic continuation procedure) and applied it to the Feynman integrals via sequential approximation (which is a finite dimensional approximation procedure). Also they have obtained the Feynman -Wiener integral and sequential Feynman integral for the functional considered in [7].

Streit and Hida[14] introduced the white noise analysis approach to the definition of the Feynman integral and D. de Falco and D. C. Khandekar[2] computed the Feynman integral for some generalized functionals.

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In Section 2, we review the basic notions of white noise analysis and some results from[4,8,10,11,13].

In Section 3, we apply the white noise analysis to the computation of Feynman integral for a generalized function, according to the definition of Feynman integrals by T. Hida and L. Streit. Moreover, we give the translation theorem for Feynman integral of a generalized function.

2. White noise distributions

We shall shortly recall some facts from white noise analysis[4, 8, 10, 13]. Let H be a real Hilbert space. Let A be a positive self-adjoint operator in H with Hilbert-Schmidt inverse. We assume that there is an orthonormal basis $\{e_j\}_{j=0}^{\infty}$ for H contained in the domain of A such that

$$Ae_j = \lambda_j e_j, \quad j = 0, 1, 2, \cdots,$$
$$1 < \lambda_0 \le \lambda_1 \le \cdots \to \infty, \quad \text{and} \quad \sum_{j=0}^{\infty} \lambda_j^{-2} < \infty.$$

For each $p \ge 0$, define

$$|\xi|_p^2 = |A^p \xi|_0^2 = \sum_{j=0}^{\infty} \lambda_j^{2p} < \xi, e_j >^2$$

and let $E_p = \{\xi \in H : |\xi|_p < \infty\}$. Then E_p is a real separable space with norm $|\cdot|_p$. It is easily seen that $E_p \subset E_q$ for any $p \ge q \ge 0$ and the inclusion map $E_{p+1} \to E_p$ is a Hilbert-Schmidt operator for any $p \ge 0$.

Let E be the projective limit of $\{E_p : p \ge 0\}$ and E^* be the topological dual space of E. Then E is a nuclear space and we get a Gel'fand triple $E \subset H \subset E^*$ with the continuous inclusions:

$$E \subset E_p \subset H \subset E_p^* \subset E^*, \quad p \ge 0,$$

where the norm of E_p^* can be checked to be given by

$$|\xi|_{-p}^2 = |A^{-p}\xi|_0^2 = \sum_{j=0}^{\infty} \lambda_j^{-2p} < \xi, e_j >^2, \quad \xi \in H$$

It is known that E^* is the inductive limit of $\{E_p^* : p > 0\}$ and that the inductive limit topology of E^* coincides with the strong dual topology. We denote by $< \cdot, \cdot >$ the canonical bilinear form $E^* \times E$.

Let μ be the standard Gaussian measure on E^*, i.e. its characteristic function is given by

$$\int_{E^*} e^{i<x,\xi>} d\mu(x) = \exp[-\frac{1}{2}|\xi|_0^2], \quad \xi \in E.$$

Then we will call (E^*, μ) *the white noise space.*

Note that for each ξ in E, the random variable $< \cdot, \xi >$ is normally distributed with mean zero and variance $|\xi|_0^2$. Obviously, this can be extened to ξ in H.

We denote by $(L^2) \equiv L^2_{\mathbb{C}}(E^*, \mu)$ the complexification of the Hilbert space of μ-square integrable functions on E^* with norm $\| \cdot \|$. By the Wiener-Ito decomposition theorem,

$$\varphi(x) = \sum_{n=0}^{\infty} <: x^{\otimes n} :, f_n > \quad x \in E^*, \quad f_n \in H_{\mathbb{C}}^{\widehat{\otimes} n} \tag{2.1}$$

where $H_{\mathbb{C}}^{\widehat{\otimes} n}$ is the n-fold symmetric tensor product of the complexification $H_{\mathbb{C}}$ of H and $: x^{\otimes n} :$ denotes the Wick ordering of $x^{\otimes n}$[10]. Moreover the (L^2)-norm $\| \varphi \|_0$ of φ is given by

$$\| \varphi \|_0^2 = \sum_{n=0}^{\infty} n! |f_n|_0^2,$$

where $| \cdot |_0$ denotes the $H_{\mathbb{C}}^{\otimes n}$-norm for any n.

Let $0 \leq \beta < 1$ be a given real number. For each $p \geq 0$, define

$$\| \varphi \|_{p,\beta}^2 = \sum_{n=0}^{\infty} (n!)^{1+\beta} |(A^p)^{\otimes n} f_n|_0^2, \quad \varphi \in (L^2)$$

where φ is given as in (2.1). Let

$$(E_p)_\beta = \{\varphi \in (L^2); \| \varphi \|_{p,\beta} < \infty\}.$$

and let $(E)_\beta$ be the projective limit of $\{(E_p)_\beta : p \geq 0\}$. Then $(E)_\beta$ is a nuclear space and we have a Gel'fand triple

$$(2.2) \qquad (E)_\beta \subset (L^2) \subset (E)^*_\beta$$

where $(E)^*_\beta$ is the topological dual space of $(E)_\beta$ and the norm on $(E_p)^*_\beta$ can be checked to be given by

$$\| \varphi \|^2_{-p,-\beta} = \sum_{n=0}^{\infty} (n!)^{1-\beta} |(A^{-p})^{\otimes n} f_n|^2_0$$

This triple is called the *Kondratiev-Streit space*[7]. If $\beta = 0$, then (2.2) is called the *Hida-Kubo-Takenaka space* and denoted by

$$(E) \subset (L^2) \subset (E)^*.$$

Furthermore we have the relationship as follows;

$$(E)_\beta \subset (E) \subset (L^2) \subset (E)^* \subset (E)^*_\beta, \quad 0 \leq \beta < 1.$$

Note that $(E)_0 = (E)$ and $(E)_{\beta_1} \subset (E)_{\beta_2}$ for any $1 > \beta_1 \geq \beta_2 \geq 0$. Moreover, for any $p \geq 0$,

$$(E_p)_\beta \subset (E_p) \subset (L^2) \subset (E_p)^* \subset (E_p)^*_\beta.$$

The elements in $(E)_\beta$ and in $(E)^*_\beta$ are called a *test function* and a *generalized function*, respectively. We denote by $\ll \cdot, \cdot \gg$ the canonical $\mathbb{C}$-bilinear form on $(E)^*_\beta \times (E)_\beta$. For each $\Phi \in (E)^*_\beta$, there exists a unique sequence $\{F_n\}_{n=0}^{\infty}, F_n \in (E^*_{\mathbb{C}})^{\widehat{\otimes} n}$ such that

$$\ll \Phi, \varphi \gg = \sum_{n=0}^{\infty} n! < F_n, f_n >, \varphi \in (E)_\beta,$$

where φ is given as in (2.1). In this case we use a formal expression for $\Phi \in (E)^*_\beta$:

$$\Phi(x) = \sum_{n=0}^{\infty} <: x^{\otimes n} :, F_n >, x \in E^*, F_n \in (E_{\mathbb{C}}^*)^{\widehat{\otimes} n}.$$

For each $\xi \in E_{\mathbb{C}}$, the function $\phi_\xi \in (E)_\beta$ given by

(2.3)
$$\phi_\xi(x) \equiv \sum_{n=0}^{\infty} <: x^{\otimes n} :, \frac{\xi^{\otimes n}}{n!} >= \exp(< x, \xi > -\frac{1}{2} < \xi, \xi >), x \in E^*$$

is called an *exponential vector.* Note that $\{\phi_\xi : \xi \in E_{\mathbb{C}}\}$ spans a dense subspace of $(E)_\beta$. The *S-transform* of $\Phi \in (E)_\beta^*$ is a function on $E_{\mathbb{C}}$ defined by

$$S\Phi(\xi) = \ll \Phi, \phi_\xi \gg, \quad \xi \in E_{\mathbb{C}} \tag{2.4}$$

For the main theorems, we need some propositions.

PROPOSITION 2.1[11]. *Let $\Phi_n \in (E)_\beta^*$ and $F_n = S\Phi_n$. Then Φ_n converges strongly in $(E)_\beta^*$ if and only if the following conditions are satisfied:*

(a) $\lim_{n\to\infty} F_n(\xi)$ exists for $\xi \in E_{\mathbb{C}}$.

(b) There exist nonnegative constants $K, a,$ and p , independent of n, such that

$$|F_n(\xi)| \le K \exp[a|\xi|_p^{\frac{2}{1-\beta}}], \quad n \in \mathbb{N}, \xi \in E_{\mathbb{C}}.$$

PROPOSITION 2.2[11]. *Let $(M, \mathcal{B}, m)$ be a measure space. Suppose a function $\Phi : M \to (S)_\beta^*$ satisfies the conditions;*

(1)$S(\Phi(\cdot))(\xi)$ is measurable for any $\xi \in E_{\mathbb{C}}$.

(2) There exist nonnegative numbers $K, a,$ and p such that

$$\int_M |S\Phi(u)(\xi)| dm(u) \le K \exp[a|\xi|_p^{\frac{2}{1-\beta}}], \quad \xi \in E_{\mathbb{C}}$$

Then Φ is pettis integrable and for any $E \in \mathcal{B}$,

$$S(\int_B \Phi(u) dm(u))(\xi) = \int_B S\Phi(u)(\xi) dm(u), \quad \xi \in E_{\mathbb{C}}$$

PROPOSITION 2.3[10]. *let $\Phi \in (E)^*$ and $y \in E$. The translation $\Phi_y(\cdot) = \Phi(\cdot - y)$ of Φ by y is also in $(E)^*$ and $S\Phi_y(\xi) = S\Phi(\xi - y), \xi \in E$.*

3. Feynman integrals in white noise analysis

Consider a nonrelativistic particle of mass $m = 1$ moving in $\mathbb{R}$ under the influence of a conservative force given by a potential V. In quantum mechanics, the state of the particle at time t is described by a wave function $\psi(t, x)$ satisfying the Schrödinger equation:

$$i\frac{\partial \psi}{\partial t} = (-\frac{1}{2}\Delta + V)\psi,$$
$$\psi(0, x) = f(x),$$

where $\int_{\mathbb{R}} |f(x)|^2 dx = 1$. If we let $\mathbb{H}$ the Hamiltonian of this particle, i.e.

$$\mathbb{H} = -\frac{1}{2}\Delta + V,$$

then we can rewrite it as

$$\frac{\partial \psi}{\partial t} = -i\mathbb{H}\psi, \quad \psi(0, x) = f(x)$$

The solution to the Schrödinger equation is given informally by

$$\psi(t, x) = \exp[-it\mathbb{H}]f(x).$$

According to Feynman[3], the above solution can be written informally as

$$\text{(3.1)} \quad \psi(t, x) = \mathcal{M} \int_{C_x} e^{\frac{i}{2}\int_0^t \dot{y}(u)^2 du - i\int_0^t V(y(u))du} f(y(t))\mathcal{D}_t^\infty(y),$$

where the integration is over the path space $C_x = \{y : [0, t] \to \mathbb{R} | y(0) = x\}$ and $\mathcal{D}_t^\infty(y)$ is a uniform measure on C_x, $\mathcal{M}$ is a normalizing constant. Now we can rewrite (3.1) as

(3.2)

$$\psi(t,x) = \mathcal{M}\int_{C_x} e^{\frac{i+1}{2}\int_0^t \dot{y}(u)^2du - i\int_0^t V(y(u))du} f(y(t)) e^{-\frac{1}{2}\int_0^t |\dot{y}(u)|^2du}\mathcal{D}_t^\infty(y),$$

In [2,11,14] they used the white noise theory to give a sense of (3.2). That is, he regarded the product of the quantities $\mathcal{M}, \mathcal{D}_t^\infty[\dot{y}]$ and $e^{-\frac{1}{2}\int_0^t \dot{y}(u)^2du}$ as a Gaussian measure on white noise space and renormalized $e^{\frac{i+1}{2}\int_0^t \dot{y}(u)^2du}$. Hence letting $y(t) = x - B(t)$ for Brownian motion $B(t)$, he rewrote (3.2) by

(3.3)

$$\psi(t,x) = \int_{S'(\mathbb{R})} (\mathcal{N}e^{\frac{i+1}{2}\int_0^t \dot{B}(u)^2du}) e^{-i\int_0^t V(x-B(u))du} f(x - B(u)) d\mu(\dot{B}),$$

where $(\mathcal{S}'(\mathbb{R}), \mu)$ is a white noise space. The integral in (3.3) can be interpreted as the evaluation of its integrand, called *Feynman integrand*, at the test functional $\varphi \equiv 1$.

Now we give the definition of Feynman integral in white niose language as follows[11,14].

Let K be a trace class operator on H such that $I + K$ is invertible and $(I + K)^{-1}K$ is a continuous linear oprator from E_p to E_q^*. If we take an abstract Wiener space (H, E_p^*) for some $p \geq \frac{1}{2}$, then for any $h \in H$ we have the equality [8]

$$\int_{E^*} \exp[-\frac{1}{2} < x, Kx >] d\mu(x) = \det(I + K)^{-\frac{1}{2}}.$$

But if K is not in trace class and is symmetric, then we need the renormalization of $\exp[-\frac{1}{2} < x, Kx >]$. We can define $\mathcal{N}\exp[-\frac{1}{2} < x, Kx >]$ by a generalized function in $(E)^*$ with $S\mathcal{N}\exp[-\frac{1}{2} < x, Kx >](\xi) = \exp[-\frac{1}{2} < \xi, (I + K)^{-1}K\xi >]$.

DEFINITION 3.1. Suppose the Feynman integrand φ in (3.3) is a generalized function in the space $(E)_\beta^*$ for some β. Then the Feynman integral $\mathcal{F}\varphi$ is defined by

$$\mathcal{F}\varphi = \ll \varphi, 1 \gg = S\varphi(0)$$

where $S\varphi$ is a S-transform of the distribution φ on a white noise space.

THEOREM 3.2. *Let K be a symmetric operator and L be a self-adjoint trass class operator on $H_{\mathbb{C}}$. Let $(I+K)$ and $(I+K+L)$ have the bounded inverses.Let $(I+K+L)^{-1}(K+L)$ be a continuous linear operator from E_p to E_q^* for some $p,q \geq 0$. Then for $h \in H_{\mathbb{C}}$ the product*

(3.5)

$$\Psi = (\mathcal{N}\exp[-\frac{1}{2} < x, Kx >])\exp[-\frac{1}{2} < x, Lx >]\exp[i < x, h >],$$

is a generalized function in $(E)^*$ with its S-transform

$$\begin{aligned} S\Psi(\xi) =& \det(I + L(I+K)^{-1})^{-\frac{1}{2}}\exp[-\frac{1}{2} < \xi, (I+K+L)^{-1}(K+L)\xi >] \\ & \exp[-\frac{1}{2} < h, (I+K+L)^{-1}h >]\exp[i < \xi, (I+K+L)^{-1}h >] \end{aligned}$$

Proof. We define a functional

$$\begin{aligned} \Psi_n =& (\prod_{k=1}^{n}(1+\alpha_k)^{\frac{1}{2}})\exp[-\frac{1}{2}\sum_{k=1}^{n}\alpha_k < x, e_k >^2] \\ & \exp[-\frac{1}{2}\sum_{k=1}^{n}\beta_k < x, e_k >^2]\exp[i < x, h >] \end{aligned}$$

where α_k's and β_k's are eigenvalues of K and L, respectively, and $\{e_k : k \geq 0\}$ is an orthonomal basis for H. Note that

$$h = \sum_{k=1}^{n} < h, e_k > e_k + h'$$

where $h' \perp e_k$ for all $1 \leq k \leq n$. Now we can write Ψ_n as follows

$$\begin{aligned} \Psi_n =& \{\prod_{k=1}^{n}(1+\alpha_k)^{\frac{1}{2}}\exp[-\frac{1}{2}(\alpha_k+\beta_k) < x, e_k >^2 \\ & + i < h, e_k >< x, e_k >]\}\exp[i < x, h' >]. \end{aligned}$$

Then we obtain the S-transform

$$\begin{aligned} S\Psi_n(\xi) &= \ll \Psi_n, \phi_\xi \gg \\ &= \ll \Psi_n, \exp < \cdot, \xi > \gg \exp[-\frac{1}{2} < \xi, \xi >] \\ &= \{\prod_{k=1}^{n} (\frac{1+\alpha_k}{1+\alpha_k+\beta_k})^{\frac{1}{2}} \exp[\frac{1}{2} \frac{< ih+\xi, e_k >^2}{1+\alpha_k+\beta_k}]\} \\ &\quad \exp[-\frac{1}{2} < \xi, \xi >] \exp[-\frac{1}{2} \parallel h' - i\xi' \parallel^2]. \end{aligned}$$

where $\xi' = \xi - \sum_{k=0}^{n} < \xi, e_k > e_k$ and ϕ_ξ is an exponential vector in (2.3). Hence for each $\xi \in E_{\mathbb{C}}$,

$$\begin{aligned} \lim_{n\to\infty} S\Psi_n(\xi) &= \det(I + L(I+K)^{-1})^{-\frac{1}{2}} \\ &\quad \exp[-\frac{1}{2} < \xi, (I+K+L)^{-1}(K+L)\xi >] \\ &\quad \exp[-\frac{1}{2} < h, (I+K+L)^{-1}h >] \\ &\quad \exp[i < \xi, (I+K+L)^{-1}h >] \end{aligned}$$

But note that for some nonnegative constants M, a, and p independent of n,

$$|S\Psi_n(\xi)| \leq M\exp[a|\xi|_p^2], \quad \xi \in E_{\mathbb{C}}.$$

since $(I+K)^{-1}$ and $(I+K+L)^{-1}$ are bounded and so $\det(I + L(I+K)^{-1})^{-\frac{1}{2}}$ exists. Hence by Proposition 2.1, Ψ_n converges strongly to a generalized function Ψ in $(E)^*$, by which we define the product in (3.5). The S- transform of Ψ is given by

$$\begin{aligned} S\Psi(\xi) &= \det(I + L(I+K)^{-1})^{-\frac{1}{2}} \\ &\quad \exp[-\frac{1}{2} < \xi, (I+K+L)^{-1}(K+L)\xi >] \\ &\quad \exp[-\frac{1}{2} < h, (I+K+L)^{-1}h >] \\ &\quad \exp[i < \xi, (I+K+L)^{-1}h >]. \end{aligned}$$

THEOREM 3.3. *Let K and L be operators as in Theorem3.2. The product*

(3.6)

$$\Phi = (\mathcal{N}\exp[-\frac{1}{2} < x, Kx >])\exp[-\frac{1}{2} < x, Lx >]\int_H \exp[i < x, h >]d\nu(h),$$

where ν is some countably additive complex measure with finite absolute variation on H, is a generalized function in $(E)^*$ with the S-transform

$$\begin{aligned} S\Phi(\xi) &= det(I + L(I + K)^{-1})^{-\frac{1}{2}} \\ &\exp[-\frac{1}{2} < \xi, (I + K + L)^{-1}(K + L)\xi >] \\ &\int_H \exp[-\frac{1}{2} < h, (I + K + L)^{-1}h >] \\ &\exp[i < \xi, (I + K + L)^{-1}h >]d\nu(h). \end{aligned} \tag{3.7}$$

Moreover, the corresponding Feynman integral is given by

$$\mathcal{F}\Phi = \det(I + L(I + K)^{-1})^{-\frac{1}{2}} \int_H \exp[-\frac{1}{2} < h, (I + K + L)^{-1}h >]d\nu(h)$$

Proof. Consider Ψ in (3.5) as the function from H to $(E)^*$, i.e. $\Psi(h) \in (E)^*$ for any $h \in H$. It is checked easily that $S(\Psi(\cdot))(\xi)$ is measurable for any $\xi \in E_{\mathbb{C}}$ and there exist nonnegative numbers M, a, and p such that

$$\int_H |S\Psi(h)(\xi)|d\nu(h) \leq M\exp[a|\xi|_p^{\frac{2}{1-\beta}}], \quad \in E_{\mathbb{C}}.$$

Hence by Proposition 2.2, we know that the integral in (3.6) is a white noise integral in $(E)^*$ in Pettis sense and

$$S(\int_H \Psi(h)d\nu(h))(\xi) = \int_H S\Psi(h)(\xi)d\nu(h), \quad \xi \in E_{\mathbb{C}}$$

In other word,

$$S\Phi(\xi) = \int_H S\Psi(h)(\xi)d\nu(\xi)$$

$$= \int_H \det(I + L(I+K)^{-1})^{-\frac{1}{2}}\exp[-\frac{1}{2} < \xi, (I+K+L)^{-1}(K+L)\xi >]$$

$$\exp[-\frac{1}{2} < h, (I+K+L)^{-1}h >]\exp[i < \xi, (I+K+L)^{-1}h >]d\nu(h)$$

$$= \det(I + L(I+K)^{-1})^{-\frac{1}{2}}\exp[-\frac{1}{2} < \xi, (I+K+L)^{-1}(K+L)\xi >]$$

$$\int_H \exp[-\frac{1}{2} < h, (I+K+L)^{-1}h >]\exp[i < \xi, (I+K+L)^{-1}h >]d\nu(h)$$

Finally we get the Feynman integral

$$\mathcal{F}\Phi = \det(I + L(I+K)^{-1})^{-\frac{1}{2}} \int_H \exp[-\frac{1}{2} < h, (I+K+L)^{-1}h >]d\nu(h).$$

THEOREM 3.4. *Let Φ be as in (3.6) and $\Phi_y(\cdot) = \Phi(\cdot + y), y \in E$. Then the Feynman integral of Φ_y is given by*

$$\mathcal{F}\Phi_y = det(I + L(I+K)^{-1})^{-\frac{1}{2}}$$

$$exp[-\frac{1}{2} < y, (I+K+L)^{-1}(K+L)y >]$$

$$\int_H exp[-\frac{1}{2} < h, (I+K+L)^{-1}h >]$$

$$exp[i < y, (I+K+L)^{-1}h >]d\nu(h)$$

and furthermore,

$$\mathcal{F}\Phi = exp[\frac{1}{2}|y|_0^2]\mathcal{F}(\Phi_y \cdot exp[< y, \cdot >])$$

or

$$\mathcal{F}\Phi_y = exp[\frac{1}{2}|y|_0^2]\mathcal{F}(\Phi \cdot exp[- < y, \cdot >])$$

Proof. By Proposition 2.3 we know that the translation Φ_y of Φ by $y \in E$ is in $(E)^*$ and $S\Phi_y(\cdot) = S\Phi(\cdot - y)$. Hence the Feynman integral of Φ_y is given by

$$\mathcal{F}\Phi_y = S\Phi_y(0) = S\Phi(-y)$$

$$= \det(I + L(I+K)^{-1})^{-\frac{1}{2}}\exp[-\frac{1}{2} < y, (I+K+L)^{-1}(K+L)y >]$$

$$\int_H \exp[-\frac{1}{2} < h, (I+K+L)^{-1}h > -i < y, (I+K+L)^{-1}h >]d\nu(h).$$

Note that the product $\Phi\phi = \phi\Phi$ of a generalized function $\Phi \in (E)^*$ and a test function $\phi \in (E)$ is defined uniquely by the formula

$$\ll \Phi\phi, \psi \gg = \ll \Phi, \phi\psi \gg, \quad \psi \in (E).$$

Now we can get

$$\begin{aligned} S(\Phi_y \exp[< y, \cdot >])(\xi) &= \ll \Phi_y, \exp[< \xi + y, \cdot >] \gg \exp[-\frac{1}{2} < \xi, \xi >] \\ &= \ll \Phi_y, \exp[< \xi + y, \cdot >] \gg \exp[-\frac{1}{2} < \xi + y, \xi + y >] \\ &\quad \exp[- < y, \xi >] \exp[-\frac{1}{2} < y, y >] \\ &= S\Phi_y(\xi + y) \exp[- < y, \xi >] \exp[-\frac{1}{2} |y|_0^2] \\ &= S\Phi(\xi) \exp[- < y, \xi >] \exp[-\frac{1}{2} |y|_0^2] \end{aligned}$$

Hence it follows that

$$\mathcal{F}\Phi = \exp[\frac{1}{2}|y|_0^2] \mathcal{F}(\Phi_y \exp[< y, \cdot >])$$

Similary we can obtain

$$\mathcal{F}\Phi_y = \exp[\frac{1}{2}|y|_0^2] \mathcal{F}(\Phi \cdot \exp[- < y, \cdot >])$$

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