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A GENERALIZATION OF COHEN–MACAULAY MODULES BY TORSION THEORY

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Abstract In this short note we study the torsion theories over a commutative ring R and discuss a relative dimension related to such theories for R-modules. Let σ be a torsion functor and (T, F) be its corresponding partition of Spec(R). The concept of σ-Cohen Macaulay (abbr. σ-CM) module is defined and some of the main points concerning the usual Cohen-Macaulay modules are extended. In particular it is shown that if M is a non-zero σ-CM module over R and S is a multiplicatively closed subset of R such that, for all minimal element of T, $S \cap \mathfrak{p} = \emptyset$, then $S^{-1}M$ is a $S^{-1}\sigma$-CM module over $S^{-1}R$, where $S^{-1}\sigma$ is the direct image of σ under the natural ring homomorphism $R \longrightarrow S^{-1}R$.

1. Introduction, notation and some properties of σ-depth

Cohen-Macaulay modules play an important role in the study of commutative algebra and some various attempts are appeared in the litrature to generalize this concept (see [5]).

Throughout this note R will denote a commutative ring with non-zero identity and σ will be a torsion functor over R. Also, (T, F) will be the corresponding partition of Spec(R) so that $T = \{\mathfrak{p} \in \text{Spec}(R) : R/\mathfrak{p} \text{ is a torsion module}\}$ and $F = \text{Spec}(R)\backslash T$. The primes in T are called torsion primes, while those in F are called torsion free primes (see [4, page, 73]). We also use T_0 to denote the set of minimal elements (primes) of T. Let R' be an another commutative ring and $\phi : R \longrightarrow R'$ be a ring homomorphism. We denote the direct image of σ under ϕ by σ_ϕ (see [4, section 3]). Let S be a multiplicatively closed subset of

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R. Then the direct image of σ under the natural homomorphism $R \longrightarrow S^{-1}R$ is denoted by $S^{-1}\sigma$. In particular if $S = R\backslash\mathfrak{p}$, for some prime ideal $\mathfrak{p}$ of R, then we denote $S^{-1}\sigma$ by $\sigma(\mathfrak{p})$.

DEFINITION 1.1. Let $(\mathcal{T}, \mathcal{F})$ be a torsion theory and M an R-module. We define the $(\mathcal{T}, \mathcal{F})$-dominant dimension of M, denoted by $(\mathcal{T}, \mathcal{F})$-$d_R(M)$, as the least integer n for which the n-th term $E^n(M)$ in a minimal injective resolution for M is not torsion free, if any such integers exist and ∞ otherwise (see[1, Definition 1.2]).

We shall see later (Corollary 1.5) that we can consider the $(\mathcal{T}, \mathcal{F})$-dominant dimension as a generalization of depth and we denote it by σ-depth, where σ is the corresponding torsion functor to the $(\mathcal{T}, \mathcal{F})$.

EXAMPLE 1.2. Let $R = \mathbb{Z}$ and G be a $\mathbb{Z}$-module. Consider the exact sequence $0 \longrightarrow G \xrightarrow{\alpha} E(G) \longrightarrow E(G)/\alpha(G) \longrightarrow 0$, where $E(G)$ is the injective envelope of G, so that $E(G)/\alpha(G)$ is an injective $\mathbb{Z}$-module. If G is a torsion free $\mathbb{Z}$-module, then $E(G)$ is a torsion free $\mathbb{Z}$-module and thus $\sigma\text{\_depth}_{\mathbb{Z}}\, G = 1$ or ∞. If G is not torsion free then $E(G)$ is not torsion free and $\sigma\text{\_depth}_{\mathbb{Z}}\, G = 0$. Hence $\sigma\text{\_depth}_{\mathbb{Z}}\, G \in \{0, 1, \infty\}$.

PROPOSITION 1.3. *Let $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ be an exact sequence of R-modules and R-homomorphisms. Then one of the following must hold:*

(i) $\sigma\text{\_depth}_R\, M' \geq \sigma\text{\_depth}_R\, M = \sigma\text{\_depth}_R\, M''$;
(ii) $\sigma\text{\_depth}_R\, M \geq \sigma\text{\_depth}_R\, M' = 1 + \sigma\text{\_depth}_R\, M''$;
(iii) $\sigma\text{\_depth}_R\, M'' \geq \sigma\text{\_depth}_R\, M = \sigma\text{\_depth}_R\, M'$;

Proof. Let $n \in \mathbb{N}_0$ ($\mathbb{N}_0$, denotes the set of non negative integers) and $\sigma\text{\_depth}_R\, M'' = n + 1$.

Case 1: $\sigma\text{\_depth}_R\, M = n + 1$. For any R-module N. We have

$$\sigma\text{\_depth}_R\, N = \inf\{i \in \mathbb{N}_0 : \mathrm{H}^{\,i}_{\sigma}(N) \neq 0\}$$

(see [2, Ch VI, Corollary 1.6]). The exact sequence $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ induces the exact sequence

$$(\star) \qquad \mathrm{H}^{\,i-1}_{\sigma}(M'') \longrightarrow \mathrm{H}^{\,i}_{\sigma}(M') \longrightarrow \mathrm{H}^{\,i}_{\sigma}(M) \longrightarrow \mathrm{H}^{\,i}_{\sigma}(M'').$$

For all $i = 0, 1, \dots, n$, $\mathrm{H}\,^i_\sigma(M) = \mathrm{H}\,^i_\sigma(M'') = 0$, thus $\mathrm{H}\,^i_\sigma(M') = 0$ for all $i = 0, 1, \dots, n$ and $\sigma\text{-}\mathrm{depth}_R\, M' \geq n+1$.

Case 2: $\sigma\text{-}\mathrm{depth}_R\, M < n+1$. By the exact sequence $(\star)$ we have $\mathrm{H}\,^i_\sigma(M) = \mathrm{H}\,^i_\sigma(M')$ for all $i = 0, 1, \dots, n$, thus $\sigma\text{-}\mathrm{depth}_R\, M = \sigma\text{-}\mathrm{depth}_R\, M'$.

Case 3: $\sigma\text{-}\mathrm{depth}_R\, M > n+1$. So that we have $\mathrm{H}\,^i_\sigma(M') = \mathrm{H}\,^{i-1}_\sigma(M'')$ for $i = 1, 2, \dots, n+1$. Thus $\mathrm{H}\,^i_\sigma(M') = 0$ for $i = 0, 1, \dots, n+1$ and since the sequence $0 \longrightarrow \mathrm{H}\,^{n+1}_\sigma(M'') \longrightarrow \mathrm{H}\,^{n+2}_\sigma(M')$ is exact then $\mathrm{H}\,^{n+2}_\sigma(M') \neq 0$, and $\sigma\text{-}\mathrm{depth}_R\, M' = n+2$.

The cases $\sigma\text{-}\mathrm{depth}_R\, M'' = 0$ or ∞ are trivial.

COROLLARY 1.4. *Let* $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ *be an exact sequence of R-modules and R-homomorphisms. Then*

$$\sigma\text{-}\mathrm{depth}_R\, M \geq \min\{\sigma\text{-}\mathrm{depth}_R\, M', \sigma\text{-}\mathrm{depth}_R\, M''\}.$$

In the remaining of this section R will be a Noetherian ring and I an ideal of R. Let $T = \mathrm{V}\,(I)$ and $(\mathcal{T}_I, \mathcal{F}_I)$ be the torsion theory corresponding to the partition (T, F) of $\mathrm{Spec}(R)$. We denote the torsion functor corresponding to $(\mathcal{T}_I, \mathcal{F}_I)$, by σ_I.

COROLLARY 1.5. *Let* $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ *be an exact sequence of finitely generated R-modules and R-homomorphisms and I an ideal of R. Then one of the following must hold:*

(i) $\mathrm{depth}_R(I, M') \geq \mathrm{depth}_R(I, M) = \mathrm{depth}_R(I, M'')$;
(ii) $\mathrm{depth}_R(I, M) \geq \mathrm{depth}_R(I, M') = 1 + \mathrm{depth}_R(I, M'')$;
(iii) $\mathrm{depth}_R(I, M'') \geq \mathrm{depth}_R(I, M) = \mathrm{depth}_R(I, M')$;

Proof. For a finitely generated R-module M, $\sigma_I\text{-}\mathrm{depth}_R\, M = \mathrm{depth}_R(I, M)$ (see [4, Theorem 2.1]). Now by Proposition 1.3 the statement is obvious.

PROPOSITION 1.6. *Let M be an R-module, and* $a_1, \dots, a_n$ *be an M-sequence in* $\bigcap_{\mathfrak{p} \in T_0} \mathfrak{p}$. *Then* $\sigma\text{-}\mathrm{depth}_R\, M \geq n$.

Proof. We show by induction on n. Let $n = 1$ and $\mathfrak{p} \in \mathrm{Ass}_R(M)$. Note that $\mathfrak{p} \notin T$, otherwise $a_1 \in \mathfrak{p}$ which is a contradiction, so that $\mathrm{Ass}_R(M) \subseteq F$. Thus by [3, Theorem 3.1] M is a torsion free R-module and by [4, Result 1.3] $\sigma\text{-depth}_R M \geq 1$. Now suppose that $n > 1$ and that the statement holds up to $n-1$. By induction hypothesis $\mathrm{H}^{\,i}_{\sigma}(M) = 0$ and $\mathrm{H}^{\,i}_{\sigma}(M/a_1M) = 0$, for $i = 0, 1, \ldots, n-2$. From the exact sequence

$$0 \longrightarrow \mathrm{H}^{\,n-1}_{\sigma}(M) \xrightarrow{a_1} \mathrm{H}^{\,n-1}_{\sigma}(M) \longrightarrow \mathrm{H}^{\,n-1}_{\sigma}(M/a_1M)$$

we have $\mathrm{H}^{\,n-1}_{\sigma}(M) = 0$, otherwise $\emptyset \neq \mathrm{Ass}_R(\mathrm{H}^{\,n-1}_{\sigma}(M)) \subseteq T$ (see [3, Proposition 1.4]). So that a_1 is a zero-divisor on $\mathrm{H}^{\,n-1}_{\sigma}(M)$ which is a contradiction. Thus $\sigma\text{-depth}_R M \geq n$.

NOTE. Proposition 1.6 shows that

$$\sigma\text{-depth}_R M \geq \sup\{n \in \mathbb{N}_0 : \text{there exist } a_1, \ldots, a_n \text{ in } \bigcap_{\mathfrak{p} \in T_0} \mathfrak{p} \text{ is } M\text{-sequence}\}$$

but to establish the equality seems to be a challenging one. In the following an special example shows that this is not true in general.

EXAMPLE. Let $R = \mathbb{Z}$ and $T = \{p\mathbb{Z} : p \text{ is a prime integer}\}$. Suppose that σ is the torsion functor corresponding to partition $(T, F = \{0\})$ of $\mathrm{Spec}(\mathbb{Z})$. By the minimal injective resolution $0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0$ we have $\sigma\text{-depth}_{\mathbb{Z}} \mathbb{Z} = 1$ but $\bigcap_{p\mathbb{Z} \in T_0} p\mathbb{Z} \subset Z_{\mathbb{Z}}(\mathbb{Z})$.

PROPOSITION 1.7. *Let M be an R-module and $a_1, \ldots, a_n$ be an M-sequence in $\bigcap_{\mathfrak{p} \in T_0} \mathfrak{p}$. Then*
$\sigma\text{-depth}_R M = n + \sigma\text{-depth}_R M/(a_1, \ldots, a_n)M$.

Proof. It is enough to give a proof for $n = 1$. We may assume that $\sigma\text{-depth}_R M < \infty$. Set $\sigma\text{-depth}_R M = k$. By Proposition 1.6, $k \geq 1$ and $\mathrm{H}^{\,i}_{\sigma}(M) = 0$ for $i = 0, 1, \ldots, k-1$. From the long exact sequence

$$\mathrm{H}^{\,i-1}_{\sigma}(M) \xrightarrow{a_1} \mathrm{H}^{\,i-1}_{\sigma}(M) \longrightarrow \mathrm{H}^{\,i-1}_{\sigma}(M/a_1M) \longrightarrow \mathrm{H}^{\,i}_{\sigma}(M)$$

we have $\mathrm{H}^{\,i}_{\sigma}(M/a_1M) = 0$ for each $i = 0, 1, \ldots, k-2$ and $\mathrm{H}^{\,k-1}_{\sigma}(M/a_1M) \neq 0$, thus $\sigma\text{-depth}_R M/a_1M = k-1$.

PROPOSITION 1.8. *Let X be an indeterminate and $\phi : R \longrightarrow R[X]$ be the inclusion map.* Then $\sigma\text{-}\mathrm{depth}_R\, R = \sigma_{\phi}\text{-}\mathrm{depth}_{R[X]}\, R[X]$.

Proof. By [4, Proposition 3.1] $\sigma\text{-}\mathrm{depth}_R\, R[X] = \sigma_{\phi}\text{-}\mathrm{depth}_{R[X]}\, R[X]$, thus it is sufficient to show that $\sigma\text{-}\mathrm{depth}_R\, R = \sigma\text{-}\mathrm{depth}_R\, R[X]$. By [2, Ch VI Lemma 2.15] $\mathrm{H}^{\,i}_{\sigma}(R[X]) = \mathrm{H}^{\,i}_{\sigma}(\oplus R) = \oplus\, \mathrm{H}^{\,i}_{\sigma}(R)$ for all $i \geq 0$, hence

$$\begin{aligned}\sigma\text{-}\mathrm{depth}_R\, R[X] &= \inf\{i \in \mathbb{N}_0 :\ \mathrm{H}^{\,i}_{\sigma}(R[X]) \neq 0\}\\ &= \inf\{i \in \mathbb{N}_0 :\ \mathrm{H}^{\,i}_{\sigma}(R) \neq 0\} = \sigma\text{-}\mathrm{depth}_R\, R.\end{aligned}$$

2. Some properties of σ-dimension

This section is devoted to study some properties of the σ-dimension on R-module.

DEFINITION 2.1. Let M be an R-module. We define the σ-dimension of M, denoted by $\sigma\text{-}\dim_R M$, to be the supremum of lenghts of chains $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n$ of prime ideals of $\mathrm{Supp}_R(M)$ for which $\mathfrak{p}_n \in T_0$ if this supremum exists, and ∞ otherwise. We put $\sigma\text{-}\dim_R M = -1$ if either $M = 0$ or $T_0 \cap \mathrm{Supp}_R(M) = \emptyset$.

As we shall see in the following, this is a natural generalization of the usual Krull dimension.

REMARK 2.2. Let M be an R-module.

(i) $\sigma\text{-}\dim_R M = \sup\{\dim_{R_{\mathfrak{p}}} M_{\mathfrak{p}} : \mathfrak{p} \in T_0\}$.

(ii) $\sigma\text{-}\dim_R M \leq \dim M$

(iii) From this definition it follows immediately that if $(R, \mathfrak{m})$ is a local ring, then $\sigma\text{-}\dim_R M = \dim M$ in which σ is the torsion functor generated by $\mathfrak{m}$.

Also if R (not necessarily local) is a ring and M an R-module such that $\dim M < \infty$ and there is at least one $\mathfrak{p} \in T_0 \cap \mathrm{Supp}_R(M)$ with $\dim_{R_{\mathfrak{p}}} M_{\mathfrak{p}} = \dim M$ then $\dim M = \sigma\text{-}\dim_R M$.

(iv) If M is a torsion R-module, then $\sigma\text{-}\dim_R M \in \{-1, 0\}$.

(v) If R is a Noetherian ring and M is a torsion free R-module, then $\sigma\text{-}\dim_R M \neq 0$.

PROPOSITION 2.3. *Let R be a Noetherian ring and $n \in \mathbb{N}_0$, if $n \leq \dim R$, then there exists a torsion theory $\sigma_n = (\mathcal{T}_n, \mathcal{F}_n)$ on R, such that σ_n-$\dim_R R = n$.*

Proof. Let $0 \leq n \leq \dim R$. Set

$$C_n = \{\mathfrak{p} \in \mathrm{Spec}(R) : ht\mathfrak{p} = n\} \text{ and}$$
$$T_n = \{\mathfrak{q} \in \mathrm{Spec}(R) : \mathfrak{p} \subseteq \mathfrak{q} \text{ for some } \mathfrak{p} \in C_n\}$$

it is clear that T_n is closed under specialization. If we set $F_n = \mathrm{Spec}(R) \backslash T_n$, then (T_n, F_n) is a partition of $\mathrm{Spec}(R)$. Let $\sigma_n = (\mathcal{T}_n, \mathcal{F}_n)$ be the corresponding torsion theory. Clearly we have $(T_n)_0 = C_n$ and σ_n-$\dim_R R = \sup\{\dim_{R_\mathfrak{p}} R_\mathfrak{p} : \mathfrak{p} \in C_n\} = n$.

PROPOSITION 2.4. *Let $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ be an exact sequence of R-modules and R-homomorphisms. Then σ-$\dim_R M$ is finite if and only if σ-$\dim_R M'$ and σ-$\dim_R M''$ are finite. In addition, if σ-$\dim_R M$ is finite, then σ-$\dim_R M = \max\{\sigma$-$\dim_R M', \sigma$-$\dim_R M''\}$.*

Proof. Let σ-$\dim_R M = n < \infty$. Thus there is $\mathfrak{p} \in T_0$ such that $\dim_{R_\mathfrak{p}} M_\mathfrak{p} = n$. For all $\mathfrak{q} \in T_0$, by the exact sequence $0 \longrightarrow M'_\mathfrak{q} \longrightarrow M_\mathfrak{q} \longrightarrow M''_\mathfrak{q} \longrightarrow 0$ we have

$$\dim_{R_\mathfrak{q}} M_\mathfrak{q} = \max\{\dim_{R_\mathfrak{q}} M'_\mathfrak{q}, \dim_{R_\mathfrak{q}} M''_\mathfrak{q}\}$$

thus $\dim_{R_\mathfrak{q}} M'_\mathfrak{q} \leq \dim_{R_\mathfrak{q}} M_\mathfrak{q} \leq \dim_{R_\mathfrak{p}} M_\mathfrak{p} = n$ and $\dim_{R_\mathfrak{q}} M''_\mathfrak{q} \leq \dim_{R_\mathfrak{q}} M_\mathfrak{q} \leq \dim_{R_\mathfrak{p}} M_\mathfrak{p} = n$ hence σ-$\dim_R M'$, σ-$\dim_R M''$ are finite and

$$\max\{\sigma\text{-}\dim_R M', \sigma\text{-}\dim_R M''\} \leq \sigma\text{-}\dim_R M.$$

Now, let σ-$\dim_R M' = m < \infty$ and σ-$\dim_R M'' = k < \infty$. For all $\mathfrak{p} \in T_0$

$$\dim_{R_\mathfrak{p}} M_\mathfrak{p} = \max\{\dim_{R_\mathfrak{p}} M'_\mathfrak{p}, \dim_{R_\mathfrak{p}} M''_\mathfrak{p}\} \leq \max\{m, k\}$$

then σ-$\dim_R M$ is finite and σ-$\dim_R M \leq \max\{m, k\}$.

PROPOSITION 2.5. *Let R be a Noetherian ring and M an R-module with $\sigma\text{-}\dim_R M = k$ $(k \in \mathbb{N})$, and let for all $\mathfrak{p} \in T_0$, $M_\mathfrak{p}$ be a finitely generated $R_\mathfrak{p}$-module. If $a_1, \ldots, a_n$ is an M-sequence in $\bigcap_{\mathfrak{p}\in T_0} \mathfrak{p}$ then*

$$\sigma\text{-}\dim_R M = n + \sigma\text{-}\dim_R M/(a_1, \ldots, a_n)M.$$

Proof. For all $\mathfrak{p} \in T_0 \cap \operatorname{Supp}_R(M)$, $(R_\mathfrak{p}, \mathfrak{p}R_\mathfrak{p})$ is a Noetherian local ring, $M_\mathfrak{p}$ is a non-zero finitely generated $R_\mathfrak{p}$-module and $a_1/1, \ldots, a_n/1$ is an $M_\mathfrak{p}$-sequence in $\mathfrak{p}R_\mathfrak{p}$. Thus $k \geq n$ and

$$\dim_{R_\mathfrak{p}} M_\mathfrak{p} = n + \dim_{R_\mathfrak{p}} M_\mathfrak{p}/(a_1/1, \ldots, a_n/1)M_\mathfrak{p}$$

(see [8, Exercise 16.1]). Also, for all $\mathfrak{p} \in T_0 \bigcap \operatorname{Supp}_R(M)$

$$\dim_{R_\mathfrak{p}} \big(M/(a_1, \ldots, a_n)M\big)_\mathfrak{p} = \dim_{R_\mathfrak{p}} M_\mathfrak{p} - n \leq k - n.$$

In particular $\dim_{R_\mathfrak{q}} M_\mathfrak{q} = k$ for some $\mathfrak{q} \in T_0$. Hence

$$\dim_{R_\mathfrak{q}} \big(M/(a_1, \ldots, a_n)M\big)_\mathfrak{q} = k - n.$$

Then $\sigma\text{-}\dim_R M/(a_1, \ldots, a_n)M = k - n$.

PROPOSITION 2.6. *Let S be a multiplicatively closed subset of R. If for all $\mathfrak{p} \in T_0$, $S \cap \mathfrak{p} = \emptyset$, then for any R-module M,*

$$\sigma\text{-}\dim_R M = S^{-1}\sigma\text{-}\dim_{S^{-1}R} S^{-1}M.$$

Proof. Let $\phi : R \longrightarrow S^{-1}R$ be the natural homomorphism, and (T^ϕ, F^ϕ) be the partition of $\operatorname{Spec}(S^{-1}R)$ corresponding to $S^{-1}\sigma$. Let $\mathfrak{p} \in \operatorname{Spec}(R)$ and $\mathfrak{p} \cap S = \emptyset$. Then by [4, page, 76], $\mathfrak{p} \in T$ if and only if $S^{-1}\mathfrak{p} \in T^\phi$. It follows that $\mathfrak{p} \in T_0$ if and only if $S^{-1}\mathfrak{p} \in (T^\phi)_0$. Now

$$\begin{aligned} &S^{-1}\sigma\text{-}\dim_{S^{-1}R} S^{-1}M \\ &\quad = \sup\left\{\dim_{(S^{-1}R)_{S^{-1}\mathfrak{p}}} (S^{-1}M)_{S^{-1}\mathfrak{p}} : S^{-1}\mathfrak{p} \in (T^\phi)_0\right\} \\ &\quad = \sup\left\{\dim_{R_\mathfrak{p}} M_\mathfrak{p} : \mathfrak{p} \in T_0\right\} = \sigma\text{-}\dim_R M. \end{aligned}$$

In the remaining of this section R will be a Notherian ring.

PROPOSITION 2.7. *Let X be an indeterminate. Then*

$$\sigma\text{-}\dim_R R = \sigma_\phi\text{-}\dim_{R[X]} R[X].$$

Proof. Let $\phi : R \longrightarrow R[X]$ be the natural homomorphism and (T^ϕ, F^ϕ) be the partition of $\mathrm{Spec}(R[X])$ corresponding to σ_ϕ. For $\mathfrak{p} \in \mathrm{Spec}(R)$, it is easy to see that $\mathfrak{p} \in T$ if and only if $\mathfrak{p}[X] \in T^\phi$. Thus if $P \in (T^\phi)_0$ then $P = P^c[X]$, where $P^c = P \cap R$. Also, it is easy to see that $P \in (T^\phi)_0$ if and only if $P^c \in T_0$. By [8, Theorem 15.1] for any $P \in \mathrm{Spec}(R[X])$

$$\text{ht } P = \text{ ht } P^c + \dim \frac{(R[X])_P}{P^c(R[X])_P}.$$

If $P \in (T^\phi)_0$, then $\dim \frac{(R[X])_P}{P^c(R[X])_P} = 0$ and $\text{ht } P = \text{ ht } P^c$. Thus

$$\begin{aligned}\sigma_\phi\text{-}\dim_{R[X]} R[X] &= \sup\{\dim(R[X])_P : P \in (T^\phi)_0\}\\ &= \sup\{\text{ ht } P : P \in (T^\phi)_0\}\\ &= \sup\{\text{ ht } P^c : P^c \in T_0\} = \sigma\text{-}\dim_R R.\end{aligned}$$

The following theorem generalizes [6, Theorem 2.2].

THEOREM 2.8. *Let M be an R-module and for all $\mathfrak{p} \in T_0$, $M_\mathfrak{p}$ be a finitely generated $R_\mathfrak{p}$-module. If $\sigma\text{-}\dim_R M = n$ $(n \in \mathbb{N}_0)$ then $H^n_\sigma(M) \neq 0$. Furthermore, if $k = \sup\{i \in \mathbb{N}_0 : H^i_\sigma(M) \neq 0\}$ and $\mathrm{Supp}_R(H^k_\sigma(M)) \cap T_0 \neq \emptyset$, then $\sigma\text{-}\dim_R M = k$.*

Proof. Let $\mathfrak{p} \in T_0$ and $\phi : R \longrightarrow R_\mathfrak{p}$ be the natural homomorphism and (T^ϕ, F^ϕ) be the partition of $\mathrm{Spec}(R_\mathfrak{p})$ corresponding to $\sigma(\mathfrak{p})$. It is easy to see that $T^\phi = \{\mathfrak{p}R_\mathfrak{p}\}$, thus $\sigma(\mathfrak{p}) = \sigma_{\mathfrak{p}R_\mathfrak{p}}$. There is $\mathfrak{p} \in \mathrm{Supp}_R(M) \cap T_0$ such that $\dim_{R_\mathfrak{p}} M_\mathfrak{p} = n$ and

$$n = \sup\{i \in \mathbb{N}_0 : \text{ H}^i_{\sigma(\mathfrak{p})}(M_\mathfrak{p}) = \text{ H}^i_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p}) \neq 0\}$$

(see [6, Theorem 2.2] and [1, Corollary 3.2]), thus by [4, Proposition 3.2] $n = \sup\{i \in \mathbb{N}_0 : (\text{ H}^i_\sigma(M))_\mathfrak{p} \neq 0\}$ and $\text{H}^n_\sigma(M) \neq 0$. Now for all $\mathfrak{q} \in \mathrm{Supp}_R(M) \cap T_0$,

$$\sup\{i \in \mathbb{N}_0 : (\text{ H}^i_\sigma(M))_\mathfrak{q} \neq 0\} = \dim_{R_\mathfrak{q}} M_\mathfrak{q} \leq \dim_{R_\mathfrak{p}} M_\mathfrak{p} = n.$$

Let $\mathfrak{q} \in \mathrm{Supp}_R(\text{ H}^k_\sigma(M)) \cap T_0$. Then $(\text{ H}^k_\sigma(M))_\mathfrak{q} \neq 0$ and $k \leq \dim_{R_\mathfrak{q}} M_\mathfrak{q} \leq n$.

EXAMPLE 2.9. Let $T \subseteq \operatorname{Max}(R)$, and M be an R-module, and for all $\mathfrak{m} \in T$, $M_{\mathfrak{m}}$ be a finitely generated $R_{\mathfrak{m}}$-module. If $\sigma\text{-}\dim_R M = n (n \in \mathbb{N}_0)$ then

$$\sigma\text{-}\dim_R M = \sup\{i \in \mathbb{N}_0 : \mathrm{H}^{\,i}_{\sigma}(M) \neq 0\}.$$

REMARK 2.10. Let $(R, \mathfrak{m})$ be a local ring and $T \neq \{\mathfrak{m}\}$. Let M be an R-module and for all $\mathfrak{p} \in T_0$, $M_{\mathfrak{p}}$ be a finitely generated $R_{\mathfrak{p}}$-module. If $\sigma\text{-}\dim_R M = n$ $(n \in \mathbb{N}_0)$ then there is a $\mathfrak{p} \in T_0 \backslash \{\mathfrak{m}\}$ such that $\dim_{R_{\mathfrak{p}}} M_{\mathfrak{p}} = n$. Thus

$$\mathrm{H}^{\,n}_{\sigma(\mathfrak{p})}(M_{\mathfrak{p}}) = (\mathrm{H}^{\,n}_{\sigma}(M))_{\mathfrak{p}} \neq 0 \quad \text{and} \quad \operatorname{Supp}_R(\mathrm{H}^{\,n}_{\sigma}(M)) \nsubseteq \{\mathfrak{m}\}.$$

This shows that $\mathrm{H}^{\,n}_{\sigma}(M)$ is not Artinian.

However, the following theorems determine the cases in which $\mathrm{H}^{\,i}_{\sigma}(M)$ is Artinian.

The next result extends [10, Theorem 3.3].

THEOREM 2.11. *Let $(R, \mathfrak{m})$ be a local ring. If for every finitely generated torsion free R-module N, $\bigcap_{\mathfrak{p} \in T_0} \mathfrak{p} \nsubseteq Z_R(N)$, then for any finitely generated R-module M, with* $\dim M = n$, *$\mathrm{H}^{\,n}_{\sigma}(M)$ is Artinian.*

Proof. We use induction on n. If $n = 0$ then $\operatorname{Supp}_R(M) = \{\mathfrak{m}\}$ and M is Artinian, thus $\sigma(M) = \mathrm{H}^{\,0}_{\sigma}(M)$ is Artinian. Now let $n \geq 1$. For all $i \geq 1$, $\mathrm{H}^{\,i}_{\sigma}(M) = \mathrm{H}^{\,i}_{\sigma}(M/\sigma(M))$ and $\dim M/\sigma(M) \leq \dim M$. If $\dim M/\sigma(M) < n$, then $\mathrm{H}^{\,n}_{\sigma}(M) = 0$. So we may assume that M is torsion free. Let $a \in \bigcap_{\mathfrak{p} \in T_0} \mathfrak{p} \backslash Z_R(M)$ and $M' = M/aM$. We have $\dim M' = n - 1$ and the exact sequence $0 \longrightarrow M \xrightarrow{a} M \longrightarrow M' \longrightarrow 0$ yields an exact sequence

$$\mathrm{H}^{\,n-1}_{\sigma}(M') \longrightarrow \mathrm{H}^{\,n}_{\sigma}(M) \xrightarrow{a} \mathrm{H}^{\,n}_{\sigma}(M).$$

By induction hypothesis $\mathrm{H}^{\,n-1}_{\sigma}(M')$ is Artinian, so $0 :_{\mathrm{H}^{\,n}_{\sigma}(M)} a$ is Artinian. Let $x \in \mathrm{H}^{\,n}_{\sigma}(M)$ then there exists an ideal I of R such that R/I is torsion module and $Ix = 0$. If $I \subseteq \mathfrak{p}$, then $\mathfrak{p} \in T$ and $a \in \mathfrak{p}$, thus $a \in r(I) = \bigcap_{I \subseteq \mathfrak{p}} \mathfrak{p}$ and there is $k \in \mathbb{N}$ such $a^k \in I$ so that $a^k x = 0$. Therefore, $\mathrm{H}^{\,n}_{\sigma}(M) = \bigcup_{k \in \mathbb{N}} 0 :_{\mathrm{H}^{\,n}_{\sigma}(M)} a^k$ and $\mathrm{H}^{\,n}_{\sigma}(M)$ is Artinian (see [9, Theorem 1.3]).

The next result extends [6, Theorem 2.1].

THEOREM 2.12. *Let T be a finite subset of Max (R), M be an R-module and for all $\mathfrak{m} \in T$, $M_{\mathfrak{m}}$ a finitely generated $R_{\mathfrak{m}}$ module. Then $H^{i}_{\sigma}(M)$ is an Artinian R-module for each $i \geq 0$.*

Proof. Let

$$0 \longrightarrow M \longrightarrow E^0(M) \longrightarrow E^1(M) \longrightarrow \dots \longrightarrow E^i(M) \longrightarrow E^{i+1}(M) \longrightarrow \dots$$

be a minimal injective resolution of M. For each $i \geq 0$

$$\sigma(E^i(M)) = \bigoplus_{\mathfrak{m} \in T} \mu^i(\mathfrak{m}, M) E(R/\mathfrak{m})$$

and for each $i \geq 0$, $\mu^i(\mathfrak{m}, M) = \mu^i(\mathfrak{m}R_{\mathfrak{m}}, M_{\mathfrak{m}})$ is finite (see [8, Theorem 18.7]). Hence $\sigma(E^i(M))$ is Artinian for each $i \geq 0$ (see [7, Proposition 3]). It follows that $\mathrm{H}^{i}_{\sigma}(M)$ is Artinian for all $i \geq 0$.

3. σ-Cohen-Macaulay modules

In this section R is Noetherian ring.

PROPOSITION 3.1. *Let M be a non-zero R-module and $T_0 \subseteq \mathrm{Supp}_R(M)$. If $\sigma(\mathfrak{p})$-$\mathrm{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}$ is finite, for all $\mathfrak{p} \in T_0$, then σ-$\mathrm{depth}_R M \leq \sigma$-$\dim_R M$.*

Proof. We may assume that σ-$\dim_R M$ is finite. For all $\mathfrak{p} \in T_0$, $M_{\mathfrak{p}}$ is a non-zero $R_{\mathfrak{p}}$-module and $\sigma(\mathfrak{p})$-$\mathrm{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}$ is finite, thus $\sigma(\mathfrak{p})$-$\mathrm{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \leq \dim_{R_{\mathfrak{p}}} M_{\mathfrak{p}}$ (see [1, Corollary 3.2]). By [4, Corollary 4.2]

$$\sigma\text{-}\mathrm{depth}_R M = \inf\{\sigma(\mathfrak{p})\text{-}\mathrm{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} : \mathfrak{p} \in T\}.$$

Thus

$$\begin{aligned} \sigma\text{-}\mathrm{depth}_R M &\leq \inf\{\sigma(\mathfrak{p})\text{-}\mathrm{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} : \mathfrak{p} \in T_0\} \\ &\leq \sup\{\sigma(\mathfrak{p})\text{-}\mathrm{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} : \mathfrak{p} \in T_0\} \\ &\leq \sup\{\dim_{R_{\mathfrak{p}}} M_{\mathfrak{p}} : \mathfrak{p} \in T_0\} = \sigma\text{-}\dim_R M. \end{aligned}$$

COROLLARY 3.2. *If M is a non-zero finitely generated R-module and $T_0 \subseteq \mathrm{Supp}_R(M)$, then $\sigma\text{-}\mathrm{depth}_R M \leq \sigma\text{-}\dim_R M$.*

Proof. Let $\mathfrak{p} \in T_0$. Then $\sigma(\mathfrak{p})\text{-}\mathrm{depth}_{R_\mathfrak{p}} M_\mathfrak{p} = \sigma_{\mathfrak{p}R_\mathfrak{p}}\text{-}\mathrm{depth}_{R_\mathfrak{p}} M_\mathfrak{p}$ $= \mathrm{depth}_{R_\mathfrak{p}} M_\mathfrak{p} < \infty$. Now the claim is obvious by Proposition 3.1.

EXAMPLE 3.3. Let k be a field and $R = k[X, Y, Z]$, where X, Y and Z are indeterminates. Set $I = (X) \cap (Y, Z)$ and $T = \mathrm{V}(I)$ so that $T_0 = \{(X), (Y, Z)\}$.

$$\sigma_I\text{-}\mathrm{depth}_R R = \inf\{\mathrm{depth}(\mathfrak{p}, R) : \mathfrak{p} \in T_0\} = 1$$

(see [4, Theorem 4.3]) and $\sigma_I\text{-}\dim_R R = \sup\{\dim R_\mathfrak{p} : \mathfrak{p} \in T_0\} =$ 2. Thus

$$\sigma_I\text{-}\mathrm{depth}_R R < \sigma_I\text{-}\dim_R R < \dim R.$$

DEFINITION 3.4. Let R be a Noetherian ring, and M be a finitely generated R-module. We say that M is a σ-Cohen- Macaulay (σ- CM) module if $T_0 \subseteq \mathrm{Supp}_R(M)$ and $\sigma\text{-}\mathrm{depth}_R M = \sigma\text{-}\dim_R M$, or if $T_0 \nsubseteq \mathrm{Supp}_R(M)$. If R itself is a σ- CM R-module we say that R is σ- CM ring.

PROPOSITION 3.5. *Let $(R, \mathfrak{m})$ be a local ring and M be a finitely generated R-module. Then M is a CM module if and only if M is a $\sigma_\mathfrak{m}$-CM module.*

Proof. We may assume that $M \neq 0$, thus $T_0 = \{\mathfrak{m}\} \subseteq \mathrm{Supp}_R(M)$;

$$\sigma_\mathfrak{m}\text{-}\dim_R M = \dim_{R_\mathfrak{m}} M_\mathfrak{m} = \dim M \quad \text{and}$$
$$\sigma_\mathfrak{m}\text{-}\mathrm{depth}_R M = \mathrm{depth}_R M.$$

PROPOSITION 3.6. *If for all ideals I of R, R is σ_I-CM, then R is CM. Conversely, if R is a CM ring then R is σ_J-CM, for all unimxed ideal J of R.*

Proof. Let I be an ideal of R. We have

$$\begin{aligned}\sigma_I\text{-}\mathrm{depth}_R R &= \mathrm{depth}(I, R) \leq htI = \inf\{ht\mathfrak{p} : \mathfrak{p} \in T_0\} \\ &\leq \sup\{ht\mathfrak{p} : \mathfrak{p} \in T_0\} = \sigma_I\text{-}\dim_R R \quad (T = \mathrm{V}(I)).\end{aligned}$$

If R is σ_I-CM, then $\mathrm{depth}(I, R) = htI$, so that R is CM. Conversely, if R is CM ring and J an unmixed ideal, then for any $\mathfrak{p}$ and $\mathfrak{q}$ in $T_0(T = \mathrm{V}(J))$, we have $ht\mathfrak{p} = ht\mathfrak{q}$ and by the above relations $\sigma_J\text{-}\mathrm{depth}_R R = \sigma_J\text{-}\dim_R R$.

PROPOSITION 3.7. *(i) If M is a finitely generated torsion R-module, then M is a σ-CM module.*

(ii) Let $\mathfrak{p} \in T_0$ and M be an R-module. If $M_\mathfrak{p}$ is a non-zero finitely generated $R_\mathfrak{p}$-module, then $M_\mathfrak{p}$ is CM if and only if $M_\mathfrak{p}$ is a $\sigma(\mathfrak{p})$-CM $R_\mathfrak{p}$-module.

Proof. (i) We may assume that $M \neq 0$ and $T_0 \subseteq \mathrm{Supp}_R(M)$. It is clear from Remark 2.2 (iv) and Corollary 3.2.

(ii) Let $\mathfrak{p} \in T_0 \cap \mathrm{Supp}_R(M)$. Then

$$\sigma(\mathfrak{p})\text{-}\mathrm{depth}_{R_\mathfrak{p}} M_\mathfrak{p} = \mathrm{depth}_{R_\mathfrak{p}} M_\mathfrak{p} \quad \text{and} \quad \sigma(\mathfrak{p})\text{-}\dim_{R_\mathfrak{p}} M_\mathfrak{p} = \dim_{R_\mathfrak{p}} M_\mathfrak{p}.$$

PROPOSITION 3.8. *Let M be a σ-CM R-module and $T_0 \subseteq \mathrm{Supp}_R(M)$. Then for all $\mathfrak{p} \in T_0$, $M_\mathfrak{p}$ is a CM $R_\mathfrak{p}$-module.*

Proof. Note that

$$\begin{aligned} \sigma\text{-}\mathrm{depth}_R M &\leq \inf\{\sigma(\mathfrak{p})\text{-}\mathrm{depth}_{R_\mathfrak{p}} M_\mathfrak{p} : \mathfrak{p} \in T_0\} \\ &= \inf\{\mathrm{depth}_{R_\mathfrak{p}} M_\mathfrak{p} : \mathfrak{p} \in T_0\} \\ &\leq \sup\{\dim_{R_\mathfrak{p}} M_\mathfrak{p} : \mathfrak{p} \in T_0\} = \sigma\text{-}\dim_R M. \end{aligned}$$

Since M is σ-CM then $\inf\{\mathrm{depth}_{R_\mathfrak{p}} M_\mathfrak{p} : \mathfrak{p} \in T_0\} = \sup\{\dim_{R_\mathfrak{p}} M_\mathfrak{p} : \mathfrak{p} \in T_0\}$ so that $\mathrm{depth}_{R_\mathfrak{p}} M_\mathfrak{p} = \dim_{R_\mathfrak{p}} M_\mathfrak{p}$, for all $\mathfrak{p} \in T_0$.

PROPOSITION 3.9. *Let M be a non-zero finitely generated R-module with $\sigma\text{-}\dim_R M < \infty$. If $a_1, \ldots, a_n$ is an M-sequence in $\bigcap_{\mathfrak{p} \in T_0} \mathfrak{p}$, then M is σ-CM if and only if $M/(a_1, \ldots, a_n)M$ is σ-CM.*

Proof. We may assume that $T_0 \subseteq \mathrm{Supp}_R(M)$, so that the result is clear by Proposition 1.7 and Proposition 2.5.

PROPOSITION 3.10. *Let S be a multiplicatively closed subset of R such that $\mathfrak{p} \cap S = \emptyset$, for all $\mathfrak{p} \in T_0$. If M is a σ-CM R-module, then $S^{-1}M$ is a $S^{-1}\sigma$-CM module over $S^{-1}R$.*

Proof. For $\mathfrak{p} \in T_0$ $(S^{-1}M)_{S^{-1}\mathfrak{p}} = M_\mathfrak{p}$, thus $T_0 \nsubseteq \mathrm{Supp}_R(M)$ if and only if $(T^\phi)_0 \nsubseteq \mathrm{Supp}_{S^{-1}R}(S^{-1}M)$. Assume that $M \neq 0$ and $T_0 \subseteq \mathrm{Supp}_R(M)$. By [4, Proposition 3.2], we have

$$\sigma\text{-}\mathrm{depth}_R M \leq S^{-1}\sigma\text{-}\mathrm{depth}_{S^{-1}R} S^{-1}M$$

and by Corollary 3.2 $S^{-1}\sigma\text{-depth}_{S^{-1}R}\, S^{-1}M \leq S^{-1}\sigma\text{-dim}_{S^{-1}R}\, S^{-1}M$. Since M is σ-CM then by Proposition 2.6

$$S^{-1}\sigma\text{-depth}_{S^{-1}R}\, S^{-1}M = S^{-1}\sigma\text{-dim}_{S^{-1}R}\, S^{-1}M.$$

PROPOSITION 3.11. *R is σ-CM if and only if $R[X]$ is σ_ϕ-CM.*

Proof. By Proposition 1.8 and Proposition 2.7 the statement is obvious.

PROPOSITION 3.12. *Let M be a finitely generated R-module with $\sigma\text{-dim}_R M = n$ $(n \in \mathbb{N}_0)$ such that $T_0 \subseteq \mathrm{Supp}_R(M)$. Set $k = \sup\{i \in \mathbb{N}_0 : H^{\,i}_{\sigma}(M) \neq 0\}$. Then M is σ-CM and $T_0 \cap \mathrm{Supp}_R(H^{\,k}_{\sigma}(M)) \neq \emptyset$ if and only if $H^{\,i}_{\sigma}(M) = 0$, for all $i \neq n$.*

Proof. By Theorem 2.8, $\mathrm{H}^{\,n}_{\sigma}(M) \neq 0$ and $n = k$. Now since M is σ-CM thus $\mathrm{H}^{\,i}_{\sigma}(M) = 0$ for all i with $i < n$. The converse is obvious.

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