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RADICALS OF SUBMODULES AND CLOSED SUBMODULES

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Abstract In this note we characterize the radical of a submodule by its invelope under some conditions and prove propositions about closed submodules.

1. Introduction

In this paper, all rings are commutative with identity,all modules are unitary and $B \leqslant A$ means that B is a submodule of the R-module A. *M-radical* of a submodule $B \leqslant A$ is the intersection of all prime submodules of A containing B. $P \leqslant A$ is a *prime submodule* provided $P \neq A$ and for $r \in R, a \in A \backslash P$, such that $ra \in P$,it follows that $rA \subseteq P$. In case $A = R$,clearly prime submodules coincide with prime ideals.

The *envelope* of $B, E_A(B)$, is the collection of all $x \in A$ which there exists $r \in R, a \in A$ such that $x = ra$ and $r^n a \in B$ for some $n \in Z^+$. The envelope of a submodule is not,in general,a submodule. We say that A *satisfies the radical formula* (A.s.t.r.f.) if for every $B \leqslant A$, the radical of B is the submodule generated by its envelope,i.e, rad $B =< E_A(B) >$. Clearly,rad $A =< E_A(A) >$. In the section 2 of this note, we prove that multiplication modules and vector spaces satisfy the radical formula(Theorem 2.6 and Proposition 2.7) In section 3, Using the concept of closure (Theorem 3.3),we find out an equivalent condition to be a closed submodule under some condiotions (Theorem 3.2 and Theorem 3.4).

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2. Radicals of Submodules

PROPOSITION 2.1. *Let $B \leqslant A$. Then $< E_A(B) > \subseteq radB$.*

Proof. Let $x \in E_A(B)$ and P a prime submodule of A containing B. Then, $x = ra, r \in R, a \in A$ and $r^n a \in B$.Thus $r^n a \in P$. Since P is a prime submodule of A, $a \in P$ or $r^n A \subseteq P$. If we note that $(P : A)$ is a prime ideal of R, then we know that $rA \subseteq P$. So, in any case, $x = ra \in P$. Hence,$E_A(B) \subseteq radB$ and so $< E_A(B) > \subseteq radB$.

PROPOSITION 2.2. *Let A and A' be R-modules with $\phi : A \to A'$ an R-module epimorphisim and $B \leqslant A$ such that $B \supseteq K = Ker\phi$. Then,$\phi(E_A(B)) = E_{A'}(\phi(B))$ and $\phi(< E_A(B) >) =< E_{A'}(\phi(B) >$.*

Proof. First,let $\phi(x) \in \phi(E_A(B))$ and $x \in E_A(B)$. Then $x = ra, r^n a \in B$ for some integer $n \in Z^+$. $\phi(x) = \phi(ra) = r\phi(a) \in A', r^n\phi(a) \in \phi(B) i.e, \phi(x) \in E_{A'}(\phi(B))$, On the other hand,if $x' \in E_{A'}(\phi(B))$,then $x' = ra'$ and $r^s a' \in \phi(B)$ for some integer $s \in Z^+, a' \in A'$ Hence,$x' = \phi(ra), \phi(r^s a) = \phi(b)$ for some $b \in B, a \in A$. $r^s a - b \in K \subseteq B$. Thus, $r^s a \in B$. Since $ra \in E_A(B), x' = \phi(E_A(B))$.

Now let $x \in \phi(< E_A(B) >)$.Then, $x = \phi(s_1x_1 + \cdots + s_kx_k)$ and $s_i \in R, x_i \in E_A(B)$. Since $\phi(E_A(B) = E_{A'}(\phi(B))$, $\phi(x_i) = r_ia'_i, r_i^{n_i}a'_i \in \phi(B)$. Hence $x = s_1\phi(x_1) + \cdots + s_k\phi(x_k) = s_1r_1a'_1 + \cdots + s_kr_ka'_k \in < E_{A'}(\phi(B)) >$. i.e, $\phi(< E_A(B) > \subseteq < E_{A'}(\phi(B)) >$

Conversely, let $y \in < E_{A'}(\phi(B) >$. Then, $y = t_1a'_1 + \cdots + t_sa'_s, t_i \in R, a'_i \in E_{A'}(\phi(B))$. Since $E_{A'}(\phi(B)) = \phi(E_A(B))$, $a'_i = \phi(r_ia_i)$ and $r_ia_i \in E_A(B)$. Thus,$y = t_1a'_1 + \cdots + t_sa'_s = t_1\phi(r_1a_1) + \cdots + t_s\phi(r_sa_s) = \phi(t_1r_1a_1) + \cdots + \phi(t_sr_sa_s) = \phi(t_1r_1a_1 + \cdots + t_sr_sa_s) \in \phi(< E_A(B) >)$. Hence $< E_{A'}(\phi(B) > \subseteq \phi(< E_A(B) >)$.

PROPOSITION 2.3. *Let A and A' be modules over a ring R and $\phi : A \to A'$ an R-module epimorphism such that $B \leqslant A$ and $B \supseteq K = Ker\phi$.*

If $rad\{B/(B : A)A\} =< E_{A/(B:A)A}(B/(B : A)A) >$ in the R-module $A/(B : A)A$, then $radB =< E_A(B) >$.

Proof. Consider an R-module epimorphism $\phi : A \to A/(B : A)A$.Clearly, $Ker\phi = (B : A)A \subseteq B$. Then we know that $radB =$

$rad\phi^{-1}(B/(B:A)A) =< E_A\{\phi^{-1}(B/(B:A)A)\} >$([4],Theorem 1.5-(ii))$=< E_A(B) >$.

PROPOSITION 2.4. *Let B be a submodule of an R-module A such that $R/(B:A)$ s.t.r.f. Then $radB =< E_A(B) >$.*

Proof. Since $R/(B:A)$ s.t.r.f, every $R/(B:A)$-module M s.t.r.f. Specially $R/(B:A)$-module $A/(B:A)A$ s.t.r.f. Therefore $rad(B/(B:A)A) =< E_{A/(B:A)A}(B/(B:A)A >$. Hence $radB =< E_A(B) >$(Proposition 2.3).

PROPOSITION 2.5. *Let R be a ring. If for every free R-module F, F s.t.r.f then R s.t.r.f.*

Proof. We know that every R-module A is the epimorphic image of a free R-module F. Hence, there exists an R-module epimorphism $\phi : F \to A$ where F is a free R-module.

For any $B \leqslant A$, there exist $F_1 \leqslant F$ such that $\phi(F_1) = B$ and $F_1 \supseteq Ker\phi$ ([4],Result 1.1). Hence $radB = rad\phi(F_1) = \phi(radF_1)$ ([4],Corollary 1.3). Since F s.t.r.f, $\phi(radF_1) = \phi(< E_F(F_1) >) =< E_A(\phi(F_1)) >$(Proposition 2.2)$=< E_A(B) >$. i.e, $radB =< E_A(B) >$.

THEOREM 2.6. *Let A be a multiplication R-module and $B \leqslant A$. Then, $< E_A(B) >= radB$,i.e, A satisfies the radical formula.*

Proof. It is enough to show that $radB \subseteq< E_A(B) >$ (Proposition 2.1).By Theorem 2.12 of [1], rad $B = rad(B:A)A$. Let $x \in radB$. Then, $x = r_1a_1 + \cdots + r_na_n$ where $r_i \in rad(B:A)$ and $a_i \in A$. Therefore there exist $n_i \in Z^+$ such that $r_i^{n_i} \in (B:A)$ and so $r_i^{n_i}a_i \in B$ for each i. Thus, $r_ia_i \in E_A(B)$. So, rad $B \subseteq< E_A(B) >$.

PROPOSITION 2.7. *Any vector space A satisfies the radical formula.*

Proof. It follows from the fact that any proper subspace of A is prime submodule ([2],Pro- position 2.1)and Proposition 2.1.

3. Closed Submodules

Let R be an integral domain,A a projective $R-$ module,and $A^* = Hom_R(A,R)$. Following[7],given $B \leqslant A$, let $B^{\perp} = \{f \in A^* | B \subseteq Ker f\}$. Also, given $C \subseteq A^*$, let $C^{\perp} = \{a \in A | f(a) = 0, \forall f \in C\} = \bigcap_{f \in C} Ker f$. In the case $B = B^{\perp\perp}$,B is said to be *closed*.

PROPOSITION 3.1. *Let R be an integral domain,A a projective R- module.If $B, C \leqslant A$ such that $B \subseteq C$, then (1) $B^{\perp} \supseteq C^{\perp}$,(2) $B^{\perp\perp} \subseteq C^{\perp\perp}$ and (3) $B^{\perp\perp}$ is a closure of B ,i.e, the smallest closed submodule containing B.*

Proof. (1) and (2) are clear. Now let's prove (3). Since $B^{\perp} = B^{\perp\perp\perp}$([7],p67), $B^{\perp\perp}$ is a closed submodule of A containing B. Let $\bar{B} = \bigcap_{B \subseteq B'} B'$ where B' is closed. Hence, $\bar{B} \subseteq B^{\perp\perp}$.

Conversely, Let B' be any closed submodule of A containing B and $a \in B^{\perp\perp}$ Then,by(2), $a \in B'^{\perp\perp} = B'$. Thus $a \in \bar{B}$.

THEOREM 3.2. *Let R be an integral domain and F a finite dimensional free R-module. If $B \lneq F$ is closed then $KB \neq KF$.*

Proof. Since B is a proper closed submodule, B is prime([4], Lemma 3.1). Therefore if $KB = KF$ then,$(B : F) \neq 0$([5], Theorem 1.7-(i)). Hence there exists $r \neq 0 \in R$ such that $rF \subseteq B$. For each $f \in B^{\perp}, rF \subseteq Ker f$. Thus $f(rF) = 0$. Since $r \neq 0$ and R is a domain, $f(F) = 0$, i.e, $f = 0$. However $B = B^{\perp\perp} = \bigcap_{f \in B^{\perp}} Ker f = ker 0 = F$, we get a contradicton. So, $KB \neq KF$.

THEOREM 3.3. *Let R be an integral domain and F a finite dimensional free R-module. Then, for $B \leqslant F$, $B^{\perp\perp} = KB \bigcap F$, where K is the quotient field of R.*

Proof. In case $KB = KF$ we show that $B^{\perp\perp} = F$ and $KB \bigcap F = F$. Hence we know that $B^{\perp\perp} = KB \bigcap F$. In fact, given $a \in F$, $a \in KF = KB$ and $a = \frac{t_1}{r_1} b_1 + \cdots + \frac{t_s}{r_s} b_s$ where $r_i \in R - 0, t_i \in R$ and $b_i \in B$. Let $\prod_{i=1}^{s} r_i = r, \hat{r_i} = r_1 \cdots r_{i-1} r_{i+1} \cdots r_s$. Hence $ra = t_1 \hat{r_1} b_1 + \cdots + t_s \hat{r_s} b_s$. For $f \in B^{\perp}, f(ra) = rf(a) = t_1 \hat{r_1} f(b_1) + \cdots + t_s \hat{r_s} f(b_s) = 0$. Since $r \neq 0, f(a) = 0. i.e, a \in \bigcap_{f \in B^{\perp}} Ker f = B^{\perp\perp}$. Thus $B^{\perp\perp} = F$.

In case $KB \neq KF$, $KB \bigcap F \neq F$ and $KB \bigcap F$ is a prime and $K(KB \bigcap F) = KB$([5],Theorem 1.9). So, $K(KB \bigcap F) \neq KF$ and $KB \bigcap F$ is closed([5],Theorem 3.9). Furthermore $KB \bigcap F$ contains B. By Proposition 3.1, $B^{\perp\perp} \subseteq KB \bigcap F$.

Conversely, if $f \in B^{\perp}$, $f : F \to R$ is an R-homomorphism. Now f can be extended to an $\tilde{f} \in Hom(KF, K)$ as follows;

$\tilde{f} : KF \to K, \tilde{f}(k_1e_1 + \cdots + k_ne_n) = k_1f(e_1) + \cdots + k_nf(e_n)$ where $\{e_1, ..., e_n\}$ is a base of F. Then $\tilde{f}|_F = f$. Thus, $\forall f \in B^{\perp}, f(B) = 0$, so $\tilde{f}(KB) = Kf(B) = 0$. Therefore $0 = \tilde{f}(KB \bigcap F)$ $= f(KB \bigcap F)$ and so $KB \bigcap F \subseteq kerf$. i.e,$KB \bigcap F \subseteq \bigcap_{f \in B^{\perp}} Kerf$ $= B^{\perp\perp}$.

Now we give a partial converse of Theorem 3.2

THEOREM 3.4. *Let R be an integral domian and F a finite dimensional free R-module If B is a primary submodule of F such that $KB \neq KF$, then B is closed.*

Proof. We know that B is prime([5],Theorem 1.7). Therefore $B = KB \bigcap F$([5],Lemma 1.8). However, $B^{\perp\perp} = KB \bigcap F$ (Theorem 3.3). Hence $B = B^{\perp\perp}$. i.e, B is closed.

References

1. Z.El-Bast and P.F.Smith, *Multiplication Modules*, Comm.in Algebra **vol 16(4)** (1988), 755-779.
2. Yong Hwan Cho, *Prime Submodules*, Bull.of Honam. Math(to appear).
3. Chin-Pi Lu, *Prime submodules of Modules*, Comment. Math. Univ. Sanct. **vol.33(1).** (1984), 61-69.
4. R.L. McCasland and M.E.Moore, *On radicals of Submodules*, Comm.in Algebra. **vol.19(5).** (1991), 1327-1341.
5. R.L.Mccasland and M.E.Moore, *Prime submodules*, Comm.in Algebra **vol 20(6)** (1992), 1803-1817.
6. R.L.Mccasland and P.F.Smith, *Prime submodules of Noetherian Modules*, Rocky Mount. Journal of Math. **vol.23(3).** (1993), 1041-1061.
7. B.R.McDonald, *Linear Algebra over Commutative Rings, Pure and Applied Mathematics,A series of Monographs and Textbooks,87,*, Marcel Dekker (1984).