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SOME REMARKS ON FOUR KINDS OF POINTS ASSOCIATED TO LIE ALGEBRAS

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Abstract Exact definitions of four kinds of points shall be defined associated to Lie algebras L over an algebraically closed field F of prime characteristic $p > 0$. Next, rough bound of dimensions for L-irreducible modules associated to subregular points shall be established by taking advantage of Premet's result.

1. Introduction

In this paper, we are mainly concerned with irreducible representations of any finite dimensional restricted Lie algebra L with some mild restriction, which shall be specified later on, over an algebraically closed field F of nonzero characteristic p.

Along the way in the sequel, we are impelled to define 4 kinds of points, namely regular point, subregular point, P-regular point and P-point from the standpoint of dimensionality as in [24], which shall be recapitulated in section 2.

Most notations are those shown up in [24] and [26]. Incidentally, let L be a finite dimensional retricted Lie algebra over F and V an irreducible nonretricted L-module, i.e., $u(L, \mathcal{X})$-module for some $\mathcal{X} \in L^*$. Furthermore, we let $\pi : \mathcal{U}(L) \to \mathfrak{gl}(V)$ be its associated representation of V. By dint of Jacobson's density

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theorem, $\mathcal{U}(L)/Ker\ \pi \cong M_n(F)$ with $n = dim_F V$ as associative algebras. Since $\mathcal{U}(L)$ is Noetherian, $Ker\ \pi$ becomes a finitely generated $\mathcal{U}(L)$-module.

We meet with 2 cases in connection with such a situation. First, there is a case when generators of $Ker\ \pi \subset \mathfrak{Z}(\mathcal{U}(L))$. Secondly, the other case may happen, i.e., generators of $Ker\ \pi \not\subset \mathfrak{Z}(\mathcal{U}(L))$. As soon as some necessary nomenclature is defined, we shall see easily that in the former case V is obtained from some subregular point in some affine space, and in the latter case V is obtained from a regular point (Proposition (5.3)).

For a Lie algebra L assumed in the first paragraph, we intend to verify that if L_s is a proper subalgebra of L and M is an irreducible $u(L, \mathcal{X})$-module obtained from a subregular point, then we have

$$p^{m'} < dimM \leq p^m - rp$$

for some $r \in \mathbb{Z}^{>0}$, where $p^{2m} = [Q(\mathcal{U}(L)) : Q(\mathfrak{Z}(\mathcal{U}(L)))]$ and $p^{m'}$ is a divisor of any irreducible $u(L_s, \mathcal{X}|_{L_s})$-module in M. This shall be established in section 4 following the support variety in section 3.

Finally, a criterion for a point to be subregular shall be fittingly made up.

2. Definitions of 4 kinds of points

Let L be a finite dimensional restricted Lie algebra over F with a basis $\{x_i | 1 \leq i \leq n\}$; let $\mathcal{O}(L)$ be the algebra generated by 1 and $\{x_i^p - x_i^{[p]}\} \cup \mathfrak{Z}(L)$, i.e., $\mathcal{O}(L) := alg_F\langle\{x_i^p - x_i^{[p]}\} \cup \mathfrak{Z}(L)\rangle$. It is easily known that $\mathcal{O}(L)$ becomes the Noether normalization of $\mathfrak{Z}(\mathcal{U}(L)) =: \mathfrak{Z}$, whence for some s_i's $(1 \leq i \leq n')$ which are integral over $\mathcal{O}(L)$, $\mathfrak{Z}(\mathcal{U}(L)) = \mathcal{O}(L)[s_1, \cdots, s_{n'}]$. Letting $h : \mathcal{O}(L)[X_1, \cdots, X_{n'}] \to \mathfrak{Z}(\mathcal{U}(L))$ be the evaluation algebra homomorphism sending X_i to s_i for $1 \leq i \leq n'$, we obtain $\mathfrak{Z}(\mathcal{U}(L)) = \mathcal{O}(L)[s_1, \cdots, s_{n'}] \cong \mathcal{O}(L)[X_1, \cdots, X_{n'}]/Ker\ h$ which is nothing but a coordinate ring on a normal algebraic variety $V(Ker\ h)$ of degree n [48]. In this context an arbitrary maximal ideal of $\mathfrak{Z}(\mathcal{U}(L))$ can be represented by a coordinate $(\xi_1, \cdots, \xi_n, \eta_1, \cdots, \eta_{n'})$, where η_i's are roots of $Ker\ h = 0$ for independent variables ξ_j's $(1 \leq j \leq n)$ corresponding to variables $x_j^p - x_j^{[p]}$.

Following Zassenhaus, we have a mapping φ going from the set of all finite dimensional irreducible L-modules onto $Spec_m(\mathfrak{Z})$ which is the set of all maximal ideals of $\mathfrak{Z} := \mathfrak{Z}(\mathcal{U}(L))$. Here shall be defined 4 kinds of points in this spectrum as follows : we call $(0, \cdots, 0, \eta_1, \cdots, \eta_{n'})$ a *P − point* since it gives rise to P−representations ; in particular, we mean, by a *regular P − point*, that it is a P-point and its associtated irreducible module has dimension p^m ; the point $(\xi_1, \cdots, \xi_n, \eta_1, \cdots, \eta_{n'})$ with $dim_F(\mathcal{U}(L)/m_s) = p^{2m}$ gives rise to a p^m-dimensional S−representation $(S \in L^* \backslash \{0\})$ [43], where m_s is just a maximal 2-sided ideal containing the ideal $\sum_{i=1}^{n} \mathcal{U}(L)(x_i^p - x_i^{[p]} - \xi_i) + \sum_{j=1}^{n'} \mathcal{U}(L)(s_j - \eta_j)$ with ξ_i's and η_j's in F satisfying $Ker\ h = 0$ if they replace $x_i^p - x_i^{[p]}$ and s_j's respectively, so that the point $(\xi_1, \cdots, \xi_n, \eta_1, \cdots, \eta_{n'})$ shall be called a *regular point* ; the rest case gives rise to S−representation $(S \in L^* \setminus \{0\})$-modules of dimension $< p^m$, so that we shall call such a point a *subregular point* [24].

3. Support variety

In this section, we shall prove the theorem (3.1) by slightly modifying Premet's proof in [26] and [27].

For a vector space V over F, $\widehat{V}$ shall denote the F-vector space with underlying additive group V and with $f \in F$ acting through multiplication by p. It is known that

$$H(L) := \bigoplus_{j \geq 0} Ext^{2j}_{u(L,0)}(F, F)^\wedge$$

turns out to be a commutative associative F-algebra by the Yoneda product [26]. It also becomes a Noetherian F-algebra by [26]. The affine algebraic variety associated with $H(L)$ shall be denoted by $|L|$. For each $\varphi \in L^*$, we have a split Lie algebra extension $F \oplus L$ with the $[p]$-mapping given by $(f, l)^{[p]} = (\varphi(l)^p, l^{[p]})\ \forall (f, l) \in F \oplus L$. Following Hochschild, the Hochschild map $L^* \to Ext^2_{u(L,0)}(F, F)^\wedge$ from this construction extends multiplicatively to a natural algebra homomorphism $h^* : S(L^*) \to H(L)$. Putting $h : |L| \to$

$Spec(L^*) \cong L$, Jantzen obtained $h(|L|) = N_p(L)$ with $N_p(L) = \{x \in L | x^{[p]} = 0\}$ [16,Satz 2.14]. For a finite dimensional $u(L, \mathcal{X})$-module M with $\mathcal{X} \in L^*$, the graded vector space

$$H_{\mathcal{X}}(M) := \bigoplus_{j \geq 0} Ext^j_{u(L,\mathcal{X})}(M, M)^\frown$$

is acted by the graded algebra $H(L)$. Denoting by J_M the annihilator in $H(L)$ of the image of id_M in $Ext_{u(L,\mathcal{X})}(M, M)^\frown$, we are informed that J_M becomes a graded ideal of $H(L)$; putting $|L|_M := \{\xi \in |L| : j(\xi) = 0 \ \forall j \in J_M\}$, we see that the morphism $h : |L| \to N_p(L)$ is finite, so that $h(|L|_M)$ becomes a Zariski closed conical subset of $N_p(L)$ [26]. Putting $V_L(M) := h(|L|_M)$, we call it *support variety* of M.

Hereafter, we let G be a semisimple and simply connected F-group with $\mathfrak{g} := Lie(G)$. Now set $Z_{\mathfrak{g}}(\mathcal{X}) := \{x \in \mathfrak{g} | \mathcal{X}([x, \mathfrak{g}]) = 0\}$. We denote by R the root system of G relative to a maximal torus $T \subset G$. Jantzen defined that p is special for G if either $p = 2$ and R has a component of type B_l, C_l for $l \geq 2$ or F_4, or $p = 3$ and R has a component of type G_2. Premet has shown the following theorem in [26]. Henceforth p is assumed to be not special for G unless otherwise specified.

THEOREM (3.1). *Let G and $\mathfrak{g}$ be as above and p not special for G. If M is a nonzero $\mathfrak{g}$-module with p-character $\mathcal{X} \in \mathfrak{g}^*$, then $V_{\mathfrak{g}}(M) \subset N_p(\mathfrak{g}) \cap Z_{\mathfrak{g}}(\mathcal{X})$.*

It is not difficult to know that in verifying this theorem, we may suppose that G is simple and simply connected. Premet showed this by taking advantage of the following propositions :

PROPOSITION (3.2). *Let $G \ncong SL(2)$ and $f \in \mathfrak{g}^* \backslash \{0\}$. Let $\mathcal{E} := (Ad\,G) \cdot e_{\widehat{\alpha}} \cup \{0\}$ for the Cartan decomposition $\mathfrak{g} = \mathfrak{t} \oplus \sum_{\alpha \in R} F e_\alpha$ with respect to a maximal torus T with the maximal root $\widehat{\alpha}$, where $\mathfrak{t} = Lie(T)$. We now define $b_f : \mathfrak{g} \times \mathfrak{g} \to F$ by $b_f(x, y) = f([x, y])^2 + 4f(x)f(y)\langle x, y\rangle$, where $<,>$ is $Ad\,G$-invariant symmetric bilinear form defined at page 240 in [26]. Then $b_f|_{\mathcal{E} \times \mathcal{E}} \neq 0$.*

PROPOSITION (3.3). *Suppose that $G \not\cong SL(2)$. Suppose further that $f \in \mathfrak{g}^*$ has a nilpotent element x of $\mathfrak{g}$ such that $f([x, \mathfrak{g}]) \neq 0$. Then there exists $e \in \mathcal{E}$ for which $f(e) = 0$ and $f([x, e]) \neq 0$.*

Premet used proposition (3.2) only to establish proposition (3.3), but it turned out that the proof of proposition (3.2) is wrong, so that he gave out the corrigenda and addenda in [27] concerning such a fact.

In the mean time, we scrutinized the proof of proposition (3.3) verbatim and found that it can be generalized more or less to the algebraic groups, under consideration in [26], including $SL(2)$. Furthermore we noticed that proposition (3.2) is not necessarily essential to prove proposition (3.3). Such a fact shall be confirmed in proposition (3.3) $'$.

PROPOSITION (3.3)$'$. *Suppose that G and $\mathfrak{g}$ are as they are in the remark preceding theorem (3.1). Suppose further that $f \in \mathfrak{g}^*$ has a nilpotent element x of $\mathfrak{g}$ such that $f([x, \mathfrak{g}]) \neq 0$ and such that $f(e) = 0$ for $e \in \mathcal{E}$ does not force that e has no h-component with $h \in \mathfrak{t}$. Then there is $e \in \mathcal{E}$ for which $f([x, e]) \neq 0$ and $f(e) = 0$.*

Proof. Put $r := dim\ \mathfrak{g}$ and $\mathcal{E}^r := \{(x_1, \cdots, x_r) | x_j \in \mathcal{E}\}$. It becomes an irreducible affine variety. We denote by $\mathfrak{F}$ the ring of regular functions on $\mathcal{E}^r$. For i, j with $1 \leq i < j \leq r$, we define $\varphi_{i,j}$ in $\mathfrak{F}$ by $\varphi_{i,j}(x_1, \cdots, x_r) = \langle x_i, x_j \rangle$. Let $V_{ij} := \{x \in \mathcal{E}^r | \varphi_{i,j}(x) \neq 0\}$ and $Y_i := \{(x_1, \cdots, x_r) \in \mathcal{E}^r | f(x_i) \neq 0\}$ for $i \leq r$. By Lemma (2.1) and Lemma (2.3) (i) in [26], these are nonempty Zariski open subsets of $\mathcal{E}^r$. We define $f_x \in \mathfrak{g}^*$ by $f_x(y) = f([x, y])$ for $y \in \mathfrak{g}$. Obviously $f_x \neq 0$ by our assumption. Suppose that the linear functions f and f_x are not linearly independent ; then for some $\lambda \in F$, $f_x = \lambda f$ so that $f((ad\, x)^r(y)) = \lambda^r f(y)$ for every $y \in \mathfrak{g}$ yielding $\lambda = 0$, which contradicts the fact that $f_x \neq 0$. So they are linearly independent. Set $\mathcal{E}^r_{reg} := \{(x_1, \cdots, x_r) \in \mathcal{E}^r | x_1, \cdots, x_r \text{ span } \mathfrak{g}\}$. Obviously $\mathcal{E}^r_{reg}$ becomes a nonempty Zariski open subset of $\mathcal{E}^r$. The variety $\mathcal{E}^r$ being irreducible, the set $E_r := \mathcal{E}^r_{reg} \cap (\cap_{i<j} V_{ij}) \cap (\cap_i Y_i)$ is nonempty. Suppose that $(e_1, \cdots, e_r) \in E_r$ and consider the $2 \times r$ matrix

$$M_f = \begin{pmatrix} f(e_1) & f(e_2) & \cdots & f(e_r) \\ f_x(e_1) & f_x(e_2) & \cdots & f_x(e_r) \end{pmatrix}.$$

Since f and f_x are linearly independent, $rk(M_f) = 2$, so that for some $i,\ j$

$$\begin{vmatrix} f(e_i) & f(e_j) \\ f([x,e_i]) & f([x,e_j]) \end{vmatrix} \neq 0.$$

We have $\langle e_i, e_j\rangle \neq 0,\ f(e_i) \neq 0,\ f(e_j) \neq 0$ from the definition of E_r. Here we may put $e_i := e_{\widehat{\alpha}}$. For some one dimensional torus $h(s) \subset T,\ (Ad\,h(s))e_r = s^{<r,\widehat{\alpha}>}e_r$ for every $r \in R$. We decompose $\mathfrak{g}$ into weight spaces relative to $Ad\,h(t)$ giving a $\mathbb{Z}$-grading $\mathfrak{g} = \mathfrak{g}(-2) \oplus \mathfrak{g}(-1) \oplus \mathfrak{g}(0) \oplus \mathfrak{g}(1) \oplus \mathfrak{g}(2)$. Obviously, $\mathfrak{g}(\pm 2) = Fe_{\pm\widehat{\alpha}}$. Since $<,>$ is $Ad\,h(s)$-invariant, $\bigoplus_{i\geq -1}\mathfrak{g}(i) = \mathfrak{g}(2)^{\perp}$. Hence $e_j = -\langle e_i, e_j\rangle e_{-\widehat{\alpha}} + u$ with $u \in \bigoplus_{i\geq -1}\mathfrak{g}(i)$. Let $e_j(s) = (Ad\,x_{\widehat{\alpha}}(s))\cdot e_j$; then $e_j(s) \in \mathcal{E}\ \ \forall s \in F$, and incidentally

$$\begin{aligned} e_j(s) &= e_j + s[e_i, e_j] - \langle e_i, e_j\rangle s^2 e_i, \\ f(e_j(s)) &= a_0 + sa_1 + s^2 a_2, \\ f_x(e_j(s)) &= b_0 + sb_1 + s^2 b_2 \end{aligned}$$

with $a_0 = f(e_j),\ a_1 = f([e_i, e_j]),\ a_2 = -\langle e_i, e_j\rangle f(e_i)$ and $b_0 = f([x, e_j]),\ b_1 = f([x,[e_i,e_j]]),\ b_2 = -\langle e_i, e_j\rangle f([x, e_i])$. So there exist nonzero $\lambda_1,\ \lambda_2 \in F$ (not necessarily distinct) satisfying $a_2\lambda_i^2 + a_1\lambda_i + a_0 = 0$ for $i = 1, 2$. Now suppose that for $i = 1, 2,\ b_2\lambda_i^2 + b_1\lambda_i + b_0 = 0$. If $b_2 \neq 0$, we have $b_0/b_2 = a_0/a_2$ (otherwise slightly modify the set E_r using nonzero functions $4f_x(e_j)\langle e_i, e_j\rangle f(e_i)^2 + 2f_x([e_i,e_j])f([e_i,e_j])f(e_i) - f_x(e_i)f([e_i,e_j])^2$ defined on $\mathcal{E}\times\mathcal{E}$ to lead to a contradiction), whence $\langle e_i, e_j\rangle\{f(e_i)f([x,e_j]) - f(e_j)f([x,e_i])\} = 0$, contradicting our choice of e_i and e_j. If $b_2 = 0$, then $b_2' := -\langle e_i, e_j\rangle f([x, e_j]) \neq 0$. In this case, we may put $e_j := e_{\widehat{\alpha}}$ instead of e_i. Then $e_i = -\langle e_j, e_i\rangle e_{-\widehat{\alpha}} + u = -\langle e_i, e_j\rangle e_{-\widehat{\alpha}} + u$ for some $u \in \bigoplus_{i\geq -1}\mathfrak{g}(i)$. Let $e_i(s) = (Ad\,x_{\widehat{\alpha}}(s))e_i$; then $e_i(s) \in \mathcal{E}\ \ \forall s \in F$, and so

$$\begin{aligned} &e_i(s) = e_i + s[e_j, e_i] - \langle e_i, e_j\rangle s^2 e_j, \\ &f(e_i(s)) = f(e_i) + sf([e_j, e_i]) - s^2\langle e_i, e_j\rangle f(e_j) = a_0' + sa_1' + s^2 a_2', \\ &\text{and } \ f_x(e_i(s)) = b_0' + sb_1' + s^2 b_2', \end{aligned}$$

where $a_0' = f(e_i),\ a_1' = f([e_j, e_i]),\ a_2' = -\langle e_i, e_j\rangle f(e_j)$ and $b_0' = f([x, e_i]),\ b_1' = f([x,[e_j,e_i]]),\ b_2' = -\langle e_i, e_j\rangle f([x, e_j])$. Proceeding as in the above case, we meet with a contradiction. Hence we may suppose that either $b_2\lambda_1^2 + b_1\lambda_1 + b_0 \neq 0$ or $b'_2\lambda_1^2 + b'_1\lambda_1 + b'_0 \neq 0$. Now put $e = e_i(\lambda_1)$ or $e_j(\lambda_1)$. Then $f(e) = 0$ and $f([x, e]) \neq 0$ as claimed.

In fact, proposition $(3.3)'$ holds in all characteristic p including $p = 0$ and special cases. Furthermore the hypothesis is always satisfied for any $f \in \mathfrak{g}^*$ having a nilpotent element x such that $f([x, \mathfrak{g}]) \neq 0$ for any $G \not\cong SL(2)$. Even for $G \cong SL(2)$, our proposition is right if f satisfies the hypothesis.

4. Revision of Premet's theorem

Now we are prepared to verify theorem (3.1). First, we need another proposition to do this.

PROPOSITION (4.1). *Suppose that* $e \in \mathcal{E} \setminus \{0\}$ *and* $n \in N_p(\mathfrak{g})$ *; then* $[n, e]^{[p]} = \langle n, e\rangle^{p-1}[n, e]$.

Proof. See Lemma (3.4) in [26].

Suppose now that $\mathcal{X} \in \mathfrak{g}^*$ and M is a finite dimensional $u(L, \mathcal{X})$-module. By the previous remark, G may be assumed to be simple. Assume that $z \in N_p(\mathfrak{g})$ and $\mathcal{X}([z, \mathfrak{g}]) \neq 0$. We contend that $u(z, \mathcal{X})$-module $M|_{u(z,\mathcal{X})}$ becomes free. Here $M|_{u(z,\mathcal{X})}$ is the vector space M as $u(z, \mathcal{X})$-module. Putting $\lambda = \mathcal{X}(z)$, we see immediately that $F[X]/(X - \lambda)^p \cong u(z, \mathcal{X})$ as associative algebras. Hence we have only to show that the endomorphism $z - \lambda \cdot id_M$ is similar to a direct sum of nilpotent Jordan blocks of order p, which is equivalent to the equality $dimKer(z - \lambda \cdot id_M) = dimM/p$ since $(z - \lambda \cdot id_M)^p = 0$. By dint of our proposition $(3.3)'$, whatever G is in our situation and $G \not\cong SL(2)$, we have $e \in \mathcal{E}$ satisfying $f(e) = 0$ and $f([e, z]) \neq 0$. Put $\langle e, z\rangle =: \nu$. Since $e^p(M) = 0$, we have that $M_{(0)} := Ker\ e$ becomes a nonzero subspace of dimension $\geq dimM/p$. We denote by $U_{(i)}$ the ith component of the standard filtration of $\mathcal{U}(\mathfrak{g})$. Let $M_{(i)} := U_{(i)} \cdot M_{(0)}$ for $i \geq 1$ with $M_{(-1)} = 0$. We denote the normalizer of Fe in $\mathfrak{g}$ by $\mathfrak{n}$. From Lemma (2.4) in [26] and the fact that $e^{[2]} = 0$ if $p = 2$, we have $[e, [ex]] = -2\langle e, x\rangle e$ for arbitrary $x \in \mathfrak{g}$, so that $[e, \mathfrak{g}] \subset \mathfrak{n}$. Hence we have $e\,U_{(i)} \subset U_{(i)}\,e + U_{(i-1)}\,\mathfrak{n} + U_{(i-1)}$ for $i \geq 1$. Next

$$e(M_{(i)}) \subset M_{(i-1)} \text{ for } i \geq 0 \tag{*}$$

because $M_{(0)}$ is $\mathfrak{n}$-invariant. By the identity

$$\begin{aligned}[e,[z,[z,e]]+2\nu z] &= -[z,[e,[e,z]]]+2\nu[e,z] \\ &= 2\nu[z,e]-2\nu[z,e]=0,\end{aligned}$$

we also get

$$[z,[z,e]] \equiv -2\nu z\,(mod\,\mathfrak{n}). \tag{**}$$

Assuming $m \in M_{(0)} \setminus \{0\}$ and $0 \leq k \leq p-1$, we contend that

$$z^k \cdot m \notin M_{(k-1)}. \tag{***}$$

It is obviously right for $k=0$. Next suppose that $k \geq 1$ and $z^{k-1} \cdot v \notin M_{(k-2)}$ for every $v \notin M_{(0)} \setminus \{0\}$. If $z^k \cdot m \in M_{(k-1)}$, we have then $e \cdot z^k \cdot m \in M_{(k-2)}$ by virtue of (*). Combining (**) and the formula $e \cdot z^k = \sum_{i=0}^{k}(-1)^i\binom{k}{i}z^{k-i}((ad\,z)^i(e))$ yields

$$kz^{k-1}([e,z]-(k-1)\nu)m \in M_{(k-2)}. \tag{****}$$

Put $A_k :=$the endomorphism $[e,z]-(k-1)\nu \cdot id_M$. Since $[e,z] \in \mathfrak{n}$, we obtain $A_k(M_{(0)}) \subset M_{(0)}$. By virtue of proposition (4.1), we have $[e,z]^p = \nu^{p-1}[e,z]+a^p \cdot id_M$ with $a = f([e,z])$. Since $(k-1)^p \equiv (k-1)(mod\ p)$, we have $A_k^p - \nu^{p-1}A_k = [e,z]^p - \nu^{p-1}[e,z] = a^p \cdot id_M$. Being $a \neq 0$, A_k becomes an invertible endomorphism of M. Hence by (****), $z^{k-1}m' \in M_{(k-2)}$ is obtained for some $m' \in M_{(0)} \setminus \{0\}$, which contradicts our assumption, so that we have verified (***) by induction on k.

Suppose now that $\{m_1, \cdots, m_n\}$ be a basis of $M_{(0)}$. Defining $S = \{z^i \cdot m_j | 0 \leq i \leq p-1,\ 1 \leq j \leq n\}$, we would like to prove that S becomes a basis of M. Supposing that S is linearly dependent, we have $m \in M_{(0)} \setminus \{0\}$ and $k \leq p-1$ satisfying $2^k \cdot m \in \sum_{i=0}^{k-1} z^i(M_{(0)})$. But then $z^k \cdot m \in M_{(k-1)}$, which contradicts to (***). So S being linearly independent, we have $dimM \leq p \cdot dimM_{(0)} \leq dimM$ the last inequality coming from $e^p \cdot M = 0$.

Hence $dim M_{(0)} = dim M/p$, so that S becomes a basis of M. Putting $\widehat{S} := \{(z - \lambda \cdot id_M)^i \cdot m_j | 0 \leq i \leq p-1,\ 1 \leq j \leq n\}$, we easily see that $\widehat{S}$ is also a basis of M as well as S, whence $Ker(z - \lambda \cdot id_M) = (z - \lambda \cdot id_M)^{p-1}(M_{(0)})$. Hence $dim\, Ker(z - \lambda \cdot id_M) = dim M/p$. Hence $M|_{u(z,\mathcal{X})}$ becomes a free $u(z, \mathcal{X})$-module, which in turn says that $z \notin V_{\mathfrak{g}}(M)$ by virtue of Lemma (3.1) in [26]. Hence we conclude that $V_{\mathfrak{g}}(M) \subset Z_{\mathfrak{g}}(\mathcal{X}) \cap N_p(\mathfrak{g})$ holds when $G \ncong SL(2)$. For the proof of the case $G \cong SL(2)$, see the proof in [26].

5. Conclusion

Now let G and $\mathfrak{g}$ be as in theorem (3.1). Premet has shown very recently that theorem (3.1) may be further generalized to the case when p is any prime in his paper [27], of which the following proposition is an immediate consequence. But in fact, [27] is redundant since proposition (3.3)$'$ does imply theorem (3.1).

PROPOSITION(5.1). *Let M be any $u(\mathfrak{g}, \mathcal{X})$-module obtained from a subregular point. Then $p | dim M$ for any prime p.*

Proof. It is informed that $V_{\mathfrak{g}}(M)$ turns out to be $V_{\mathfrak{g}}(M) = \{x \in M |\ M|_{u(x,\mathcal{X})} \text{ is free } \}$. So assuming the contrary, we have then $V_{\mathfrak{g}}(M) = N_p(\mathfrak{g})$, so that $Z_{\mathfrak{g}}(\mathcal{X}) \subset N_p(\mathfrak{g})$ by [27]. But then $\forall \alpha \in R$, $e_\alpha^{[p]} = 0$ implies that $\mathcal{X}[e_\alpha, \mathfrak{g}] = 0$. Since $\mathcal{X} \not\equiv 0$, there exists $h_\alpha \in \mathfrak{t}$ such that $\mathcal{X}(h_\alpha) \neq 0$. But then for this $\alpha \in R$, $\mathcal{X}[e_\alpha, e_{-\alpha}] = \mathcal{X}(h_\alpha) \neq 0$, so that $\mathcal{X}[e_\alpha, \mathfrak{g}] \neq 0$ implying $e_\alpha \notin Z_{\mathfrak{g}}(\mathcal{X})$ contradicting to $e_\alpha \in N_p(\mathfrak{g})$.

Now any irreducible $u(L, \mathcal{X})$-module V is isomorphic to $\mathcal{U}(L)/\mathfrak{m}$ for some left maximal ideal $\mathfrak{m}$ of $\mathcal{U}(L)$, where L is any finite dimensional restricted Lie algebra over F. Since $\mathcal{U}(L)$ is Noetherian, $\mathfrak{m}$ becomes a finitely generated left ideal, so there exists r :=minimal number of generators of $\mathfrak{m}$ in $\mathfrak{m} \setminus \mathfrak{Z}(\mathcal{U}(L))$.

PROPOSITION(5.2). *Let L be any finite dimensional restricted Lie algebra and L_s any proper sub-Lie algebra contained in L. Let V be any irreducible $u(L, \mathcal{X})$-module with $x \in L_s$ such that $x \cdot V \neq 0$ and $x = [y, z]$ for some $y \in L \setminus L_s$. Let $p^{m'}$ be the divisor* *of any irreducible $u(L_s, \mathcal{X}|_{L_s})$-module in V for $\mathcal{X}|_{L_s} \in L_s^* \setminus \{0\}$. Then we have the inequalities :*

(i) $p^{m'} < dimV \leq p^m - p^{m'}$

(ii) $p^{m'} < dimV \leq p^m - rp$

(iii) $2p \leq dimV \leq p^m - 2p$.

Proof. We first note that V can't be an irreducible $u(L_s, \mathcal{X}|_{L_s})$-module by virtue of [25]. So the first inequality (i) is trivial. Let $\{x_1, \cdots, x_r\}$ be the set of minimal generators of $\mathfrak{m}$ in $\mathfrak{m} \setminus \mathfrak{Z}(\mathcal{U}(L))$. Then noting the tower of subspaces

$$\begin{aligned} 0 &\subset \mathfrak{Z}(\mathcal{U}(L)) \cap \mathfrak{m} \subset \{\mathfrak{Z}(\mathcal{U}(L)) \cap \mathfrak{m}\} + x_1\mathcal{U}(L) \subset \cdots \\ &\subset \{\mathfrak{Z}(\mathcal{U}(L)) \cap \mathfrak{m}\} + x_1\mathcal{U}(L) + \cdots + x_r\mathcal{U}(L) = \mathfrak{m}, \end{aligned}$$

(ii) is also straightforward. Since there is only a p-dimensional non-restricted $\mathfrak{sl}_2(F)$-module, (iii) is also straightforward.

Finally we close this paper presenting some signal proposition related to subregular points. Let L be any finite dimensional restricted Lie algebra over F and V be an irreducible $u(L, \mathcal{X})$-module for some $\mathcal{X} \in L^* \setminus \{0\}$. Let $\pi : \mathcal{U}(L) \to \mathfrak{gl}(V)$ be the associated representation of $u(L, \mathcal{X})$-module V. Then $\mathcal{U}(L)/Ker\ \pi \cong M_n(F)$ as F-algebras with $n = dim_F V$ by Jacobson's density theorem and $Ker\ \pi$ is finitely generated since $\mathcal{U}(L)$ becomes Noetherian.

PROPOSITION(5.3). *Retaining notations as above, we have :*

(i) Generators of $Ker\,\pi \not\subset \mathfrak{Z}(\mathcal{U}(L))$ if and only if V is obtained from some subregular point in some affine space.

(ii) Generators of $Ker\,\pi \subset \mathfrak{Z}(\mathcal{U}(L))$ if and only if V is obtained from a regular point.

Proof. We first observe the identity

$$\begin{aligned} p^{dim\ L} &= [Q(\mathcal{U}(L)) : Q(\mathcal{O}(L))] \\ &= [Q(\mathcal{U}(L)) : Q(\mathfrak{Z})]\,[Q(\mathfrak{Z}) : Q(\mathcal{O}(L))] \\ &= p^{2m} \times p^{dim\ L - 2m}. \end{aligned}$$

So if generators of $Ker\ \pi \subset \mathfrak{Z}(\mathcal{U}(L)) =: \mathfrak{Z}$, then there exists a set $\{y_1, y_2, \cdots, y_{p^{2m}}\} \subset \mathcal{U}(L)$ satisfying $(\mathfrak{Z}y_1 + \cdots + \mathfrak{Z}y_{p^{2m}}) \otimes_{\mathfrak{Z}} F \cong$

$\mathcal{U}(L) \otimes_{3} F \cong M_{p^m}(F)$, the last being a matrix algebra of degree p^m. The converse is also right from $[Q(\mathcal{U}(L)) : Q(3)] = p^{2m}$, so (ii) is immediate. On the other hand, (i) is gotten by contrapositive to (ii).

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