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Notes on the Sequential Yeh-Feynman Integral

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수열 Yeh-Feynman 적분에 관하여

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We shall extend the concept of the sequential Feynman integral over the Yeh-Winner space.
We shall show that prove the existence of the sequential Yeh-Feynman integral for all element of Banach algebras.

수열 파인만적분을 Yeh-Wiener 공간에서 확장하고 바나하(Banach) 대수의 모든 원에 대한 수열 Yeh-Feynman 적분의 존재성을 밝힐 것이다.

Keywords : Banach algebra, Analytic Yeh-Winner Integral, Analytic Yeh-Feynman Integral, Sequential Yeh-Feynman Integral.

I. Preliminaries

Let $C_2 \equiv C_2(Q)$ be the Yeh-Wiener space (or two parameter Wiener space) on $Q=[a,b]\times[c,d]$, that is, the space of real valued continuous functions $x(s,t)$ on Q such that $x(s,\cdot)=x(\cdot,t)=0$ for all $(s,t)\in Q$. And let $C_2^{\nu}\equiv C_2^{\nu}(Q)=X_{\alpha=1}^{\nu}C_2(Q)$ be the ν-dimensional Yeh-Wiener space.

Definition 1.1

Let F be a functional on $C_\nu^2(Q)$ which is s-a.e. defined such that the Yeh-Wiener integral

$$J(X)=\int_{C_2^{\nu}} F(\lambda^{-\frac{1}{2}}\vec{x})dm_y(\vec{x})$$

exists for all real $\lambda > 0$. If there exists a function $J^*(\lambda)$ analytic in the half plane Re $\lambda>0$ $(C^+=\{\lambda\in C: Re\lambda>0\})$ such that $J^*(\lambda)=J(\lambda)$ for all real $\lambda>0$, then $J^*(\lambda)$ is defined to be the analytic Yeh-Wiener integral of F over $C_2^{\nu}(Q)$ with parameter λ, and for $\lambda\in C^+$, we write

$$\int_{C_2^{\nu}}^{anyw_\lambda} F(\vec{x})dm_y(\vec{x})=J^*(\lambda)$$

Definition 1.2

Let q be a non-zero real parameter let F be a functional whose analytic Yeh-Wiener integral exists for $\lambda \in C+$. Then if the following limit exists, we call it the analytic Yeh-Feynman integral of F over $C_2^{\nu}(Q)$ with parameter q, and we write.

$$\int_{C_2^{\nu}}^{anyf_q} F(\vec{x})dm_y(\vec{x}) = \lim_{\substack{x \to iq \\ Re\,\lambda > 0}} \int_{C_2^{\nu}}^{anyw_\lambda} F(\vec{x})dm_y(\vec{x})$$

where λ approaches $-iq$ through C^+.

We define a σ-algebra α in $L_2^{\nu}(Q)$ and a class of complex measures m on $L_2^{\nu}(Q)$. Let α be the σ-algebra of subsets of $L_2^{\nu}(Q)$ generated by the class of sets of the form

$$\left\{\vec{\nu} \in L_2^{\nu}(Q) : \int_Q \nu_j(s,t)\,\varphi_j(s,t)\,dsdt < \lambda_j\right\} \quad \text{for } j = 1, 2, \cdots, \nu$$

where $\varphi = (\varphi_1, \cdots, \varphi_\nu)$ ranges over all elements of $L_2^{\nu}(Q)$, and $\vec{\lambda} = (\lambda_1, \cdots, \lambda_\nu)$ ranges over R^{ν}. The σ-algebra α is actually the Borel class of $L_2^{\nu}(Q)$, that is, the σ-algebra $\beta(L_2^{\nu}(Q))$ generated by the norm of open subsets of $L_2^{\nu}(Q)$.

And let $m \equiv m(L_2^{\nu}(Q))$ be the collection of complex measures of finite variation defined on $L_2^{\nu}(Q)$ with α as its σ-algebra of measurable sets. If $\mu \in m$, we set $\|\mu\| = var\,\mu$ over $L_2^{\nu}(Q)$. It is clear that m is a linear space of measures.

Definition 1.3

Let $S(\nu)$ be the space of functionals F expressible in the form.

$$F(\vec{x}) = \int_{C_2^{\nu}} \exp\left\{ i \sum_{j=1}^{\nu} \int_Q v_j(s,t)\,\tilde{d}x_j(s,t)\right\} d\mu(\vec{v})$$

for $s-a.e.\ \vec{x} \in C_2^{\nu}(Q)$, $where\ \mu \in m$.

Theorem 1.1

Let $\mu \in m(L_2^{\nu}(Q))$ and let $F \in S(\nu)$ be the stochastic Fourier transformation of μ, that is

$$F(\vec{x}) = \int_{L_2^{\nu}} \exp\left\{ i \sum_{j=1}^{\nu} \int_Q v_j(s,t)\,\tilde{d}x_j(s,t)\right\} d\mu(\vec{v})$$

Then F is analytic Yeh-Feynman integrable on $C_2^{\nu}(Q)$. If q is a non-zero real number, then

$$\int_{C_2^{\nu}}^{anyf_q} F(\vec{x})dm_y(\vec{x}) = \int_{L_2^{\nu}} \left\{ \frac{1}{zqi} \sum_{j=1}^{\nu} \int_Q (v_j(s,t))^2 dsdt \right\} d\mu(\vec{v})$$

Proof For λ we can write

$$\begin{aligned} J(\lambda) &\equiv \int_{C_2^{\nu}} F(\lambda^{-\frac{1}{2}}\vec{x})dm_y(\vec{x}) \\ &= \int_{C_2^{\nu}} \int_{L_2^{\nu}} \exp\left\{ i \sum_{j=1}^{\nu} \int_Q v_j(s,t)\,\tilde{d}(\lambda^{-\frac{1}{2}} x_j)(s,t)\right\} d\mu(\vec{v})dm_y(\vec{x}) \\ &= \int_{L_2^{\nu}} \int_{C_2^{\nu}} \exp\left\{ i\lambda^{-\frac{1}{2}} \sum_{j=1}^{\nu} \int_Q v_j(s,t)\,\tilde{d}x_j(s,t)\right\} dm_y(\vec{x})d\mu(\vec{v}) \end{aligned}$$

by the linearity of the P.W.Z. integral and the Fubini theorem.

$$\int_R \exp\{-(\alpha z^2 + \beta z)\}dz = \sqrt{\frac{\pi}{\alpha}} \exp\left(-\frac{\beta^2}{4\alpha}\right)$$

for $\alpha > 0$ and real or imaginary β, we can have

$$J(\lambda)=\frac{1}{\sqrt{2\pi}}\int_{L_2^\nu}\int_{\infty}^{\infty}\exp\left\{i\lambda^{-\frac{1}{2}}u\cdot\|\vec{v}\|_2\right\}\exp\left(-\frac{u^2}{2}\right)du\,d\mu(\vec{v})$$

$$=\int_{L_2^\nu}\exp\left\{-\frac{\|\vec{v}\|_2^2}{2\lambda}\right\}d\mu(\vec{v})$$

First note that the Yeh-Wiener integral $J(\lambda)$ exists for Re $\lambda>0$ ($\lambda\neq 0$) since the integrand is bounded by 1 and $\mu\in m$. And also note that by the Morera's theorem, the Fubini theorem, and the Cauchy integral theorem, the integral $J(\lambda)$ is analytic in $C^+=\lambda\in C:Re\lambda>0$.

Thus we have for Re $\lambda>0$.

$$\int_{C_2^\nu}^{anyw_\lambda} F(\vec{x})dm_y(\vec{x})=\int_{L_2^\nu}\exp\left\{-\frac{\|\vec{v}\|_2^2}{2\lambda}\right\}d\mu(\vec{v})$$

By the application of the Dominated Convergence Theorem we have

$$\int_{C_2^\nu}^{anyf_q} F(\vec{x})dm_y(\vec{x})=\int_{L_2^\nu}\exp\left\{-\frac{\|\vec{v}\|_2^2}{2\lambda}\right\}d\mu(\vec{v})$$

$$=\int_{L_2^\nu}\exp\left\{\frac{1}{2qi}\sum_{j=1}^{\nu}\int_Q[v_j(s,t)]^2 dsdt\right\}d\mu(\vec{v})$$

II. The Sequential Yeh-Feynman Integral over $C_2^\nu(Q)$

Let a subdivision σ of Q be given ;

$$\sigma\,; a=s_0<s_1<\cdots<s_l=b,\ c=t_0<t_1<\cdots<t_m=a$$

A be a vector of real numbers given by $A=(A^1,\cdots,A^\nu)$

where $A^\alpha=\{a^\alpha_{j,k}\}$ for $\alpha=1,2,\cdots,\nu$, j=1, 2, ⋯, l, k=1, 2, ⋯, m.

And let $\vec{x}_\sigma\equiv\vec{x}_\sigma((s,t),A)$ be a quadratic function in $C_2^\nu(Q)$ based on a subdivision σ and the vector of real numbers A, and defined by

$$\vec{x}_\sigma\equiv\vec{x}_\sigma((s,t),A)=[x^1_\sigma((s,t),A^1),\cdots,x^\nu_\sigma((s,t),A^\nu)]$$

where

$$x^\nu_\sigma((s,t),A)$$
$$=\frac{a^\alpha_{j,k}-a^\alpha_{j-1,k}-a^\alpha_{j,k-1}+a^\alpha_{j-1,k-1}}{(s_j-s_{j-1})(t_k-t_{k-1})}(s-s_{j-1})(t-t_{k-1})$$
$$=\frac{a^\alpha_{j,k-1}-a^\alpha_{j-1,k-1}}{s_j-s_{j-1}}(s-s_{j-1})+\frac{a^\alpha_{j-1,k}-a^\alpha_{j-1,k-1}}{t_k-t_{k-1}}(t-t_{k-1})+a^\alpha_{j-1,k-1}\ldots(1)$$

When $(s,t)\in[s_{j-1},s_j]\times[t_{k-1},t_k]$, $a^\alpha_{j,o}=a^\alpha_{o,k}=0$ for $\alpha=1,2,\cdots,\nu$, $j=1,2,\cdots,l$, and $k=1,2,\cdots,m$.

Remark

As $A=\{a^\alpha_{j,k}\}$ ranges over all of ν lm-dimensional real space, the quadratic funtions $((s,t),A)$ range over all quadratic approximations in $C_2^\nu(Q)$ based on σ.

Specifically, if $\vec{x}$ is a particular element of $C_2^\nu(Q)$ and if we set $a^\alpha_{j,k}=x^\alpha(S_j,t_k)$. then the function $\vec{x_\sigma}((s,t),A)$ is the quardratic approximation of $\vec{x}$ based on the subdivision σ.

Definition 2.1

Let q be a given non-zero real number and let $F(\vec{x})$ be a functional defined on a subset of $C_2^\nu(Q)$ containing all the quadratic elements of $C_2^\nu(Q)$. Let $\{\sigma_n\}$ be a sequence of subdivisions of Q such that norm $\|\sigma_n\|\to 0$, and let $\{\lambda_n\}$ be a sequence in $C^+ = \{\lambda \in C: Re\lambda > 0\}$ such that $\lambda_n \to -iq$.

Then if the integral in the right of (5, 2) exists for all n and if the following limit exists and is independent of the choice of sequences $\{\sigma_n\}$ and $\{\lambda_n\}$, we say that the sequential Yeh-Feynman integral over $C_2^\nu(Q)$ with parameter q exists and is given by

$$\int^{syf_q} F(\vec{x})d\vec{x} = \lim_{n\to\infty} r\sigma_{n,}\lambda_n \int_{R^{\nu(lm)}} \exp\left\{-\frac{\lambda_n}{2}\int_Q \left\| \frac{\partial^2 \vec{x}\sigma_n}{\partial s \partial t}((s,t),A)\right\|^2 dsdt\right\} F(\vec{x}\sigma_n((\cdot,\cdot),A))dA \ldots (2)$$

Where

$$r_{\sigma,\lambda} \equiv \left(\frac{\lambda}{2\pi}\right)^{\frac{\nu lm}{2}} \left[\prod_{j=1}^{l}\prod_{k=1}^{m}(s_j - s_{j-1})(t_k - t_{k-1})\right]^{-\frac{\nu}{2}}$$

and $A = (A^1 \cdots, A^\nu)$, $A^\alpha = (A_1^\alpha \cdots, A_m^\alpha)$

and $A_k^\alpha = (\alpha_{1,k}^\alpha \cdots, \alpha_{l,k}^\alpha)$

for $\alpha = 1, \cdots, \nu$, $k = 1, \cdots, m$

We note that l, m depend on σ and lm is the number of subrectiongles in σ, We empha size that the Lebesgue integral on the right of (2) exists for all n.

We write $\int^{sgf_q} F(\vec{x})d\vec{x}$ in stead of $\int^{sgf_q} F(\vec{x})dm_y(\vec{x})$

Let

$$W_\lambda(\sigma, A) = r_{\sigma,\lambda}\exp\left\{-\frac{\lambda}{2}\int_Q \left\|\frac{\partial^2 \vec{x}_\sigma}{\partial s\partial t}((s,t),A)\right\|^2 dsdt\right\}$$

$$= \left(\frac{\lambda}{2\Pi}\right)^{\frac{\nu lm}{2}} \left[\prod_{j=1}^{l}\prod_{k=1}^{m}(s_j - s_{j-1})(t_k - t_{k-1})\right]^{-\frac{\nu}{2}} \cdot \exp\left\{-\frac{\lambda}{2}\sum_{j=1}^{l}\sum_{k=1}^{m}\frac{\|\vec{\alpha}_{j,k} - \vec{\alpha}_{j-1,k}}{(s_j - s_{j-1})(t_k - t_{k-1})} \frac{-\vec{\alpha}_{j,k-1} + \vec{\alpha}_{j-1,k-1}\|^2}{(s_j - s_{j-1})(t_k - t_{k-1})}\right\}$$

By the notation $\lambda^{\frac{\nu lm}{2}}$ we mean $(\sqrt{\lambda})^{\nu lm}$ where Re $\sqrt{\lambda} > 0$, and $\vec{\alpha}_{j,k}$ is the vector $[\alpha_{j,k}^1, \cdots, \alpha_{j,k}^\nu]$ and $\|\vec{\alpha}_{j,k}\| = \sum_{\alpha=1}^{\nu}(\alpha_{j,k}^\nu)^2$.

Thus in terms of W, The sequential Yeh-Feynman integral defined in (2) can be written

$$\int^{syf_q} F(\vec{x})d\vec{x} = \lim_{n\to\infty}\int_{R^{\nu(lm)}} W_{\lambda_n}(\sigma_n, A) F(\vec{x}\sigma_n((\cdot,\cdot),A))dA$$

Remark

Since $\{\sigma_n\}$ and $\{\lambda_n\}$ were chosen arbitrarily and independently in the definition 1, the single limit may also be expressed as a double limit, thus

$$\int^{syf_q} F(\vec{x})d\vec{x} = \lim_{\substack{n\to\infty\\k\to\infty}} I_{n,k}$$

where

$$I_{n,k} = \int_{R^{\nu(lm)}} W_{\lambda n}(\sigma_k, A)F(\vec{x}_{\sigma_k}((\cdot,\cdot),A))dA$$

Now we introduce the Banach algebra $\hat{S}, S^*$ and relationship among some Banach algebras without proof.

Let $D_2 \approx D_2(Q)$ be the class of elements $x \in C_2(Q)$ such that $x \in AC(Q)$ and $\frac{\partial^2 x(s,t)}{\partial s \partial t} \in L_2(Q)$ where $L_2 = L_2(Q)$ is a hilbert space of Lebesgue measurable, real valued and square integrable functionals on Q and let $D_2^\nu \equiv D_2^\nu(Q) = X_{j=1}^{\nu} D_2(Q)$.

Definition 2.2

The functional F defined on a subset of $C_2^\nu(Q)$ that contains $D_2^\nu(Q)$ is said to be an element of $\hat{S}(\nu) \equiv \hat{S}(L_2^\nu)$ if there exists a measure $M \in m$ such that for $\vec{x} \in D_2^\nu(Q)$

$$F(\vec{x}) = \int_{L_2^\nu} \exp\left\{ i \sum_{j=1}^{\nu} \int_Q v_j(s,t) \frac{\partial^2 x(s,t)}{\partial s \partial t} dsdt \right\} d\mu(\vec{v})$$

We note, that if $F(\vec{x}) = G(\vec{x})$ for $s-a.e.\ \vec{x}$ in $C_2^\nu(Q)$ and for every $\vec{x}$ in $D_2^\nu(Q)$, we write $F \approx G$.

We have that $v \in L_2(Q)$, $x \in D_2(Q)$ then

$$\int_Q v(s,t)\, \tilde{d}x(s,t) = \int_Q v(s,t) \frac{\partial^2 x(s,t)}{\partial s \partial t} dsdt$$

Thus if $v \in L_2(Q)$ and $\{\varphi_n\}$, $\{\psi_n\}$ are two $C.O.N.$ sequences of $BU(Q)$, then

$$^{(\varphi_n)}\int_Q v(s,t)\, \tilde{d}x(s,t) = {}^{(\psi_n)}\int_Q v(s,t)\, \tilde{d}x(s,t)$$

for $x \in D_2(Q)$, and hence

$$\int_{L_2} \exp\left\{ i^{(\varphi_n)} \int_Q v(s,t)\, \tilde{d}x(s,t) \right\}$$
$$d\mu(\nu) x \int_{L_2} \exp\left\{ i^{(\psi_n)} \int_Q v(s,t)\, \tilde{d}x(s,t) \right\} d\mu(o)$$

Definition 2.3

Let $S^*(\nu) \equiv S^*(L_2^\nu)$ be the space of functionals F expressible in the form

$$F(\vec{x}) x \int_{L_2^\nu} \exp\left\{ i \sum_{j=1}^{\nu} \int_Q v_j(s,t)\, \tilde{d}x_j(s,t) \right\} d\mu(\vec{v})$$

where $\mu \in m$

Let $v \in L_2(Q)$ and let σ be any subdivision

$\sigma : a = S_0 < S_1 < \cdots < S_l = b$, $c = t_0 < t_1 < \cdots < t_m = b$.

We define the averaged function $v_\sigma(s,t)$ for v on σ by

$$v_\sigma(s,t) = \left\{ \frac{1}{(s_j - s_{j-1})(t_k - t_{k-1})} \int_{s_{j-1}}^{s_j} \int_{t_{k-1}}^{t_k} v(p,q)\, dpdq \right.$$

when $(s,t) \in [s_{j-1}, s_j] \times [t_{k-1}, t_k]$

for $j = l, \cdots, l$, $k = 1, \cdots, m$ when $s = b$ or $t = a$

Where there is a sequence for subdivisions $\sigma_1, \sigma_2, \cdots$, then σ, l, m, s_j and t_R will be replaced by $\sigma_n, l_n, m_n, s_{n,j}$ and $t_{n,k}$ respectively.

For any $v \in L_2(Q)$

$$\lim_{h,k \to 0} \frac{1}{hk} \int_s^{s+h} \int_t^{t+k} v(p,q)\, dpdq = v(s,t)$$

for almost everywhere (s, t) in Q.

Theorem 2.1

If $F \in S^*(\nu)$ and q is a nonzero real number, then F is sequentially Yeh-Feynman integrable and its sequential Yeh-Feynman integral is equal to its analytic Yeh-Feynman integral, that is

$$\int^{syf_q} F(\vec{x}) d\vec{x} = \int_{L_2^\nu} \exp\left\{ \frac{1}{2qi} \sum_{a=1}^{\nu} \int_Q [v_a(s,t)]^2 dsdt \right\} d\mu(\vec{v})$$

Proof

Since $F \in S^*(v)$, there exists a measur $\mu \in M$ such that

$$F(\vec{x}) \approx \int_{L_2^\nu} \exp\left\{ i \sum_{a=1}^{\nu} \int_Q [v_a(s,t) dx_\sigma(s,t) \right\} d\mu(\vec{v})$$

In particular, this equality holds for all quadratic functions $\vec{x}_{\sigma n}$. Let $\{\sigma_n\}$ and $\{\lambda_n\}$ be sequences of subdivisions of Q, a sequence in $\lambda \in C \mid Re\lambda > 0$ such that the norm $\|\sigma_n\| \to 0$, $Re\,\lambda_n > 0$ and $\lambda_n \to iq$.

$$J_n \equiv \int_{R^{\nu l_n m_n}} W_{\lambda_n}(\sigma_n, A) F(\vec{x}_{\sigma_n}((\cdot,\cdot), A))dA$$

$$\int_{R^{\nu l_n m_n}} W_{\lambda_n}(\sigma_n, A) \int_{L_2^\nu} \exp\left\{ i \sum_{a=1}^{\nu} \int_Q V_\sigma(s,t)\, dx^a_{\sigma n} ((s,t),A)\right\} d\mu(\vec{v})dA.$$

By the Fubini theorem and the properlies Paley-Wiener-Zygmand (P, W, Z) integral, we have

$$J_n \equiv \int_{L_2^\nu} \int_{R^{\nu l_n m_n}} W_{\lambda_n}(\sigma_n, A) \exp\left\{ i \sum_{a=1}^{\nu} \int_Q V_\sigma(s,t) \frac{\partial^2 x^a_{\sigma_n}((s,t),A)}{\partial s \partial t}\, dsdt\right\} dAd\mu(\vec{v})$$

Since $S^*(v)$ is a Banach algebra

$$J_n = r_{\sigma n,\lambda n} \int_{L_2^\nu} \int_{R^{\nu l_n m_n}} \exp\left\{ -\frac{\lambda_n}{2} \sum_{j=1}^{l_n} \sum_{k=1}^{m_n} \frac{\| \vec{a}_{j\cdot k} - \vec{a}_{j-1\cdot k} - \vec{a}_{j,k-1} + \vec{a}_{j-1,k-1} \|}{(s_{n,j} - s_{n,j-1})(t_{n,k} - t_{n,k-1})} \right.$$
$$\left. \cdot \exp\left\{ \sum_{a=1}^{\nu} \sum_{j=1}^{l_n} \sum_{k=1}^{m_n} \int_{s_{n,j-1}}^{s_{n,j}} \int_{t_{n,k-1}}^{t_{n,k}} v(s,t) \frac{a^a_{j\cdot k} - a^a_{j-1\cdot k} - a^a_{j,k-1} + a^a_{j-1,k-1}}{(s_{n,j} - s_{n,j-1})(t_{n,k} - t_{n,k-1})} dsdt\right\} dAd\mu(\vec{v}) \right.$$

$$r_{\sigma n,\lambda n} \int_{L_2^\nu} \int_{R^{\nu l_n m_n}} \exp\left\{ -\frac{\lambda_n}{2} \sum_{j=1}^{l_n} \sum_{k=1}^{m_n} \frac{\sum_{a=1}^{\nu} (a^a_{j,k} - a^a_{j-1,k} - a^a_{j,k-1} + a^a_{j-1,k-1})}{(s_{n,j} - s_{n,j-1})(t_{n,k} - t_{n,k-1})} \right.$$
$$\left. \cdot \exp\left\{ i \sum_{a=1}^{\nu} \sum_{j=1}^{l_n} \sum_{k=1}^{m_n} \int_{s_{n,j-1}}^{s_{n,j}} \int_{t_{n,k-1}}^{t_{n,k}} v(s,t) \frac{a^a_{j\cdot k} - a^a_{j-1\cdot k} - a^a_{j,k-1} + a^a_{j-1,k-1}}{(s_{n,j} - s_{n,j-1})(t_{n,k} - t_{n,k-1})} dsdt\right\} dAd\mu(\vec{v}) \right.$$

Let $B = \{b^a_{j,k}\}$ where
$b^a_{j,k} = a^a_{j\cdot k} - a^a_{j-1\cdot k} - a^a_{j,k-1} + a^a_{j-1,k-1}$.
Then

$$J_n = r_{\sigma n,\lambda n} \int_{L_2^\nu} \int_{R^{\nu l_n m_n}} \exp\left\{ -\frac{\lambda_n}{2} \sum_{j=1}^{l_n} \sum_{k=1}^{m_n} \sum_{a=1}^{\nu} \frac{(b^a_{j,k})^2}{(s_{n,j} - s_{n,j-1})(t_{n,k} - t_{n,k-1})} \right\}$$
$$\cdot \exp\left\{ i \sum_{a=1}^{\nu} \sum_{j=1}^{l_n} \sum_{k=1}^{m_n} \int_{s_{n,j-1}}^{s_{n,j}} \int_{t_{n,k-1}}^{t_{n,k}} v_a(s,t) \frac{b^a_{j,k}}{(s_{n,j} - s_{n,j-1})(t_{n,k} - t_{n,k-1})} dsdt\right\} dBd\mu(\vec{v})$$

$$= r_{\sigma n,\lambda n} \int_{L_2^\nu} \int_{R^{\nu l_n m_n}} \prod_{j=1}^{l_n} \prod_{k=1}^{m_n} \prod_{a=1}^{\nu} \exp\left\{ -\frac{\lambda_n (b^a_{j,k})^2}{2(s_{n,j} - s_{n,j-1})(t_{n,k} - t_{n,k-1})} + i\left(\int_{s_{n,j-1}}^{s_{n,j}} \int_{t_{n,k-1}}^{t_{n,k}} v_a(s,t)dsdt \right) \frac{v^a_{j,k}}{(s_{n,j} - s_{n,j-1})(t_{n,k} - t_{n,k-1})} \right\} dBd\mu(\vec{v})$$

$$= r_{\sigma n,\lambda n} \int_{L_2^\nu} \prod_{j=1}^{l_n} \prod_{k=1}^{m_n} \prod_{a=1}^{\nu} \left[\int_R \exp\left\{ -\frac{\lambda_n (b^a_{j,k})^2}{2(s_{n,j} - s_{n,j-1})(t_{n,k} - t_{n,k-1})} + i\left(\int_{s_{n,j-1}}^{s_{n,j}} \int_{t_{n,k-1}}^{t_{n,k}} v_a(s,t)dsdt \right) \frac{b^a_{j,k}}{(s_{n,j} - s_{n,j-1})(t_{n,k} - t_{n,k-1})} \right\} db^a_{j,k} \right] d\mu(\vec{v})$$

Since

$$\int_R \exp\left\{ -\frac{\lambda_n (b^a_{j,k})^2}{2(s_{n,j} - s_{n,j-1})(t_{n,k} - t_{n,k-1})} + i\left(\int_{s_{n,j-1}}^{s_{n,j}} \int_{t_{n,k-1}}^{t_{n,k}} v_a(s,t)dsdt \right) \frac{b^a_{j,k}}{(s_{n,j} - s_{n,j-1})(t_{n,k} - t_{n,k-1})} \right\} db^a_{j,k}$$

$$= \left[\frac{2\Pi(s_{n,j}-s_{n,j-1})(t_{n,k}-t_{n,k-1})}{\lambda_n}\right]^{\frac{1}{2}}$$

$$\exp\left\{-\frac{\left[\int_{s_{n,j-1}}^{s_{n,j}}\int_{t_{n,k-1}}^{t_{n,k}} v_\alpha(s,t)dsdt\right]^2}{2\lambda_n(s_{n,j}-s_{n,j-1})(t_{n,k}-t_{n,k-1})}\right\}$$

we have

$$J_n = r_{\sigma n,\lambda n}\int_{L_2^\nu}\prod_{j=1}^{l_n}\prod_{k=1}^{m_n}\prod_{\alpha=1}^{\nu}$$

$$\left[\frac{2\Pi(s_{n,j}-s_{n,j-1})(t_{n,k}-t_{n,k-1})}{\lambda_n}\right]^{\frac{1}{2}}$$

$$\exp\left\{-\frac{\left[\int_{s_{n,j-1}}^{s_{n,j}}\int_{t_{n,k-1}}^{t_{n,k}} v_\alpha(s,t)dsdt\right]^2}{2\lambda_n(s_{n,j}-s_{n,j-1})(t_{n,k}-t_{n,k-1})}\right\}d\mu(\vec{v})$$

And

$$r_{\sigma n,\lambda n}\equiv\left(\frac{\lambda_n}{2\Pi}\right)^{\frac{\nu l_n m_n}{2}}$$

$$\left[\prod_{j=1}^{l_n}\prod_{k=1}^{m_n}(s_{n,j}-s_{n,j-1})(t_{n,k}-t_{n,k-1})\right]^{\frac{\nu}{2}}$$

$$=\left[\prod_{j=1}^{l_n}\prod_{k=1}^{m_n}\frac{\lambda_n}{2\Pi(s_{n,j}-s_{n,j-1})(t_{n,k}-t_{n,k-1})}\right]^{\frac{\nu}{2}}$$

$$=\prod_{j=1}^{l_n}\prod_{k=1}^{m_n}\prod_{\alpha=1}^{\nu}\left[\frac{\lambda_n}{2\Pi(s_{n,j}-s_{n,j-1})(t_{n,k}-t_{n,k-1})}\right]^{\frac{1}{2}}$$

then

$$J_n=\int_{L_2^\nu}\exp\left\{-\sum_{\alpha=1}^{\nu}\sum_{j=1}^{l_n}\sum_{k=1}^{m_n}\frac{\left[\int_{s_{n,j-1}}^{s_{n,j}}\int_{t_{n,k-1}}^{t_{n,k}} v_\alpha(s,t)dsdt\right]^2}{2\lambda_n(s_{n,j}-s_{n,j-1})(t_{n,k}-t_{n,k-1})}\right\}d\mu(\vec{v})$$

We set

$$V_{\alpha,n}(s,t) = \frac{1}{(s_{n,j}-s_{n,j-1})(t_{n,k}-t_{n,k-1})}\int_{s_{n,j-1}}^{s_{n,j}}\int_{t_{n,k-1}}^{t_{n,k}} v_\alpha(p,q)dpdq$$

where $(s,t)\in[s_{n,j-1}-s_{n,j}]\times[t_{n,k}-t_{n,k-1}]$ for $j=1,\cdots,l_n, R=1,\cdots,m_n,$

and $v_{\alpha,n}(\cdot,t)=v_{\alpha,n}(s,\cdot)=0$

$$\int_Q\{v_\alpha,n(s,t)\}^2dsdt=$$

$$\sum_{j=1}^{l_n}\sum_{k=1}^{m_n}\int_{s_{n,j-1}}^{s_{n,j}}\int_{t_{n,k-1}}^{t_{n,k}}\{v_{\alpha,n}(s,t)\}^2dsdt$$

$$=\sum_{j=1}^{l_n}\sum_{k=1}^{m_n}\frac{1}{(s_{n,j}-s_{n,j-1})(t_{n,k}-t_{n,k-1})}\left\{\int_{s_{n,j-1}}^{s_{n,j}}\int_{t_{n,k-1}}^{t_{n,k}} v_\alpha(s,t)dsdt\right\}^2$$

$$\therefore J_n=$$

$$\int_{L_2^\nu}\exp\left\{-\frac{1}{2\lambda_n}\sum_{\alpha=1}^{\nu}\int_Q(v_{\alpha,n}(s,t))^2dsdt\right\}d\mu(\vec{v})$$

$$\int^{syf_q}F(\vec{x})d\vec{x}\equiv\lim_{n\to\infty}J_n$$

$$=\int_{L_2^\nu}\exp\left\{-\frac{1}{2\lambda_n}\sum_{\alpha=1}^{\nu}\int_Q(v_\alpha(s,t))^2dsdt\right\}d\mu(\vec{v})$$

We hold the following theorem by the same way of theorem 2.1

Theorem 2.2

If $F\in\hat{S}$, and q is nonzero real number, let

$$F(\vec{x})=$$

$$\int_{L_2^\nu}\exp\left\{i\sum_{\alpha=1}^{\nu}\int_Q v_j(s,t)\frac{\partial^2 x^2}{\partial s\partial t}dsdt\right\}d\mu(\vec{v})$$

where $\mu\in M$, then we have that F is sequential Yeh-Feynman integrable over $C_2^\nu(Q)$ and

$$\int^{syf_q}F(\vec{x})d\vec{x}=$$

$$\int_{L_2^\nu}\exp\left\{\frac{1}{2qi}\sum_{\alpha=1}^{\nu}\int_Q(v_\alpha(s,t))^2dsdt\right\}d\mu(\vec{v})$$

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