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ON MCSHANE-STIELTJES INTEGRAL

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Abstract. In this paper, we define the McShane-Stieltjes integral for real-valued function and prove some of its properties.

1. Introduction and preliminaries

In the late 1960's, E. J. McShane proved that the Lebesgue integral is indeed equivalent to a modified version of the Henstock integral. He broadened the class of tagged partition by not insisting that the tag of an interval belong to the interval. Since this increases of partitions subordinate to a given δ, it is more difficult for a function to be integrable in this sense. In fact, a function is McShane integrable if and only if its absolute value is McShane integrable [1]. As a result of this property, the McShane integral is equivalent to the Lebesgue integral. It is clear that every McShane integrable function is Henstock integrable and that the integrals are equal. A Riemann integrable function is McShane integrable [1], and a McShane integrable function is Pettis integrable [2].

In this paper, we shall introduce the McShane-Stieltjes integral for real-valued function which is the generalization of the McShane integral, and study its basic properties,

We begin with some terminology notations.

Definition 1.1. Let $\delta(\cdot)$ be a positive function defined on the interval $[a, b]$. A tagged interval $(x, [c, d])$ consists of an interval $[c, d] \subseteq [a, b]$

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and a point $x \in [a, b]$. The tagged interval $(x, [c, d])$ is subordinate to δ if

$$[c, d] \subseteq (x - \delta(x), x + \delta(x)).$$

The letter $\mathcal{P}$ will be used to denote a finite collection of non-overlapping tagged intervals. Let $\mathcal{P} = \{(x_i, [c_i, d_i]) : 1 \leq i \leq n\}$ be such a collection in $[a, b]$. We adopt the following terminology.

(1) The points $\{x_i\}$ are the tags of $\mathcal{P}$ and the intervals $\{[c_i, d_i]\}$ are the intervals of $\mathcal{P}$,
(2) If $(x_i, [c_i, d_i])$ is subordinate to δ for each i, then $\mathcal{P}$ is subordinate to δ,
(3) If $\mathcal{P}$ is subordinate to δ and $[a, b] = \cup_{i=1}^{n} [c_i, d_i]$, then $\mathcal{P}$ is a free tagged partition of $[a, b]$ that is subordinate to δ.

Let $\mathcal{P} = \{(x_i, [c_i, d_i] : 1 \leq i \leq n\}$ be a finite collection of non-overlapping tagged intervals in $[a, b]$, let $f : [a, b] \to R$, let F be a function defined on the subintervals of $[a, b]$ and let α be an increasing function on $[a, b]$. We will use the following notation:

$$f_\alpha(\mathcal{P}) = \sum_{i=1}^{n} f(x_i)(\alpha(d_i) - \alpha(c_i))$$

2. McShane-Stieltjes integral

We now present the definition of the McShane-Stieltjes integral. It is clear that every McShane-Stieltjes integrable function is McShane integrable.

DEFINITION 2.1. Let α be an increasing function on $[a, b]$. A function $f : [a, b] \to R$ is McShane-Stieltjes integrable with respect to α on $[a, b]$ if there exists a real number L with the following property : for each $\epsilon > 0$, there exists a positive function δ on $[a, b]$ such that $|f_\alpha(\mathcal{P}) - L| < \epsilon$ whenever $\mathcal{P}$ is a free tagged partition of $[a, b]$ that is subordinate to δ. The function f is McShane-Stieltjes integrable on a measurable set $E \subseteq [a, b]$ with respect to α if f_{χ_E} is McShane - Stieltjes integrable with respect to α on $[a, b]$.

Theorem 2.2. *Let α be an increasing function on $[a,b]$ and let $f=0$ nearly everywhere on $[a,b]$. If $\alpha \in C^1[a,b]$, then f is McShane-Stieltjes integrable with respect to α on $[a,b]$ and $\int_a^b f d\alpha = 0$.*

Proof. Let $\{a_n | n \in Z^+\} = \{x \in [a,b] : f(x) \neq 0\}$ and let $\epsilon > 0$. Since $\alpha \in C^1[a,b]$, there exists M such that $|\alpha'(x)| \leq M$ for all $x \in [a,b]$. By the mean-value theorem, there exists $x_{i_0} \in (c_i, d_i)$ such that

$$\alpha(d_i) - \alpha(c_i) = \alpha'(x_{i_0})(d_i - c_i).$$

Define a positive function δ by

$$\delta(x) = \begin{cases} 1 & \text{if } x \in [a,b] - \{a_n | n \in Z^+\} \\ \frac{\epsilon}{|f(a_n)|M2^{n+1}} & \text{if } x = a_n. \end{cases}$$

Suppose that $\mathcal{P} = \{(x_i, [c_i, d_i]) : 1 \leq i \leq q\}$ is a free tagged partition of $[a,b]$ that is subordinate to δ and assume that each tag occurs only once. Let π be the set of all indices i such that $x_i \in \{a_n : n \in Z^+\}$ and for each $i \in \pi$, choose n_i so that $x_i = a_{n_i}$. Then

$$\begin{aligned} |f_\alpha(\mathcal{P})| &= |\sum_{i\in\pi} f(x_i)[\alpha(d_i) - \alpha(c_i)]| \\ &= |\sum_{i\in\pi} f(x_i)\alpha'(x_{i_0})(d_i - c_i)| \\ &\leq \sum_{i\in\pi} 2|f(a_{n_i})|M\delta(a_{n_i}) \leq \sum_{i\in\pi} \epsilon 2^{-n_i} < \epsilon \end{aligned}$$

Hence, the function f is McShane-Stieltjes integrable with respect to α on $[a,b]$ and $\int_a^b f d\alpha = 0$. □ □

We next verify the basic properties of the McShane-Stieltjes integral. Just as in the McShane integral, there is a Cauchy criterion for function to be McShane-Stieltjes integrable. This is the content of the next Theorem.

Theorem 2.3. *Let α be an increasing function on $[a,b]$. A function $f : [a,b] \to R$ is McShane-Stieltjes integrable with respect to α on $[a,b]$ if and only if for each $\epsilon > 0$ there exists a positive function δ on $[a,b]$ such that $|f_\alpha(\mathcal{P}_1) - f_\alpha(\mathcal{P}_2)| < \epsilon$ whenever $\mathcal{P}_1$ and $\mathcal{P}_2$ are free tagged partitions of $[a,b]$ that are subordinate to δ.*

Proof. Suppose first that f is McShane-Stieltjes integrable with respect to α on $[a,b]$ and $\epsilon > 0$. There exists a positive function δ on $[a,b]$ such that

$$|f_\alpha(\mathcal{P}_1) - \int_a^b f d\alpha| < \frac{\epsilon}{2}, \quad |f_\alpha(\mathcal{P}_2) - \int_a^b f d\alpha| < \frac{\epsilon}{2}$$

whenever $\mathcal{P}_1$ and $\mathcal{P}_2$ are free tagged partitions of $[a,b]$ that are subordinate to δ. Then

$$\begin{aligned} |f_\alpha(\mathcal{P}_1) - f_\alpha(\mathcal{P}_2)| &\le |f_\alpha(\mathcal{P}_1) - \int_a^b f d\alpha| + |f_\alpha(\mathcal{P}_2) - \int_a^b f d\alpha| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

Hence, the Cauchy criterion is satisfied. Conversely, suppose that the Cauchy criterion is satisfied. For each positive integer n, choose a positive function δ_n on $[a,b]$ such that

$$|f_\alpha(\mathcal{P}_1) - f_\alpha(\mathcal{P}_2)| < \frac{1}{n}$$

whenever $\mathcal{P}_1$ and $\mathcal{P}_2$ are free tagged partitions of $[a,b]$ that are subordinate to δ_n. We may assume that the sequence $\{\delta_n\}$ is nondecreasing. For each n, let $\mathcal{P}_n$ be a free tagged partition of $[a,b]$ that is subordinate to δ_n. The sequence $\{f_\alpha(\mathcal{P}_n)\}$ is a Cauchy sequence since

$$m > n \ge N \quad \text{implies} \quad |f_\alpha(\mathcal{P}_m) - f_\alpha(\mathcal{P}_n)| < \frac{1}{N}$$

Let L be the limit of this sequence and let $\epsilon > 0$. Choose a positive integer N such that

$$\frac{1}{N} < \frac{\epsilon}{2} \quad \text{and} \quad |f_\alpha(\mathcal{P}_n) - L| < \frac{\epsilon}{2} \quad \text{for all } n \ge N$$

Let $\mathcal{P}$ be a free tagged partition of $[a,b]$ that is subordinate to δ_N on $[a,b]$ and compute

$$|f_\alpha(\mathcal{P}) - L| \le |f_\alpha(\mathcal{P}) - f_\alpha(\mathcal{P}_N)| + |f_\alpha(\mathcal{P}_N) - L| < \frac{1}{N} + \frac{\epsilon}{2} < \epsilon$$

Hence, the function f is McShane-Stieltjes integrable with respect to α on $[a,b]$. □ □

THEOREM 2.4. *Let α be an increasing function on $[a,b]$. Let $f : [a,b] \to R$. If f is McShane-Stieltjes integrable with respect to α on $[a,b]$, then f is McShane-Stieltjes integrable with respect to α on every subinterval of $[a,b]$.*

Proof. Let $[c,d] \subseteq [a,b]$ and let $\epsilon > 0$. Choose a positive function δ on $[a,b]$ such that $|f_\alpha(\mathcal{P}_1) - f_\alpha(\mathcal{P}_2)| < \epsilon$ whenever $\mathcal{P}_1$ and $\mathcal{P}_2$ are free tagged partitions of $[a,b]$ that are subordinate to δ. Fix free tagged partitions $\mathcal{P}_a$ of $[a,c]$ and $\mathcal{P}_b$ of $[d,b]$ that are subordinate to δ. Let $\mathcal{P}_1'$ and $\mathcal{P}_2'$ be free tagged partitions of $[c,d]$ that are subordinate to δ and define $\mathcal{P}_1 = \mathcal{P}_a \cup \mathcal{P}_1' \cup \mathcal{P}_b$ and $\mathcal{P}_2 = \mathcal{P}_a \cup \mathcal{P}_2' \cup \mathcal{P}_b$. Then $\mathcal{P}_1$ and $\mathcal{P}_2$ are free tagged partitions of $[a,b]$ that are subordinate to δ and

$$|f_\alpha(\mathcal{P}_1') - f_\alpha(\mathcal{P}_2')| = |f_\alpha(\mathcal{P}_1) - f_\alpha(\mathcal{P}_2)| < \epsilon.$$

Hence, the function f is McShane-Stieltjes integrable with respect to α on every subinterval of $[a,b]$. □ □

THEOREM 2.5. *Let α be an increasing function on $[a,b]$. Let $f : [a,b] \to R$ and let $c \in (a,b)$. If f is McShane-Stieltjes integrable with respect to α on each of the intervals $[a,c]$ and $[c,b]$, then f is McShane- Stieltjes integrable with respect to α on $[a,b]$ and $\int_a^b f d\alpha = \int_a^c f d\alpha + \int_c^b f d\alpha$.*

Proof. Let $\epsilon > 0$. By hypothesis, there exists a positive function δ_1 on $[a,c]$ such that $|f_\alpha(\mathcal{P}) - \int_a^c f d\alpha| < \frac{\epsilon}{2}$ whenever $\mathcal{P}$ is a free tagged partition of $[a,c]$ that is subordinate to δ_1 and a positive function δ_2 on $[c,b]$ such that $|f_\alpha(\mathcal{P}) - \int_c^b f d\alpha| < \frac{\epsilon}{2}$ whenever $\mathcal{P}$ is a free tagged partition of $[c,d]$ that is subordinate to δ_2. Define δ on $[a,b]$ by

$$\delta(x) = \begin{cases} \min\{\delta_1(x), c-x\}, & \text{if } a \le x < c; \\ \min\{\delta_1(c), \delta_2(c)\}, & \text{if } x = c; \\ \min\{\delta_2(x), x-c\}, & \text{if } c < x \le b. \end{cases}$$

Let $\mathcal{P}$ be a free tagged partition of $[a,b]$ that is subordinate to δ and suppose that each tag occurs only once. Note that $\mathcal{P}$ must be of the form $\mathcal{P}_a \cup (c,[u,v]) \cup \mathcal{P}_b$ where the tags of $\mathcal{P}_a$ are less than c and the tags of $\mathcal{P}_b$ are greater than c. Let $\mathcal{P}_1 = \mathcal{P}_a \cup (c, [u, c])$ and let $\mathcal{P}_2 = \mathcal{P}_b \cup (c, [c, v])$. Then $\mathcal{P}_1$ is a free tagged partition of $[a, c]$ that is subordinate to δ_1 and $\mathcal{P}_2$ is a free tagged partition of $[c, b]$ that is subordinate to δ_2. Since $f_\alpha(\mathcal{P}) = f_\alpha(\mathcal{P}_1) + f_\alpha(\mathcal{P}_2)$, we obtain

$$\begin{aligned}|f_\alpha(\mathcal{P}) - &\int_a^c f d\alpha - \int_c^b f d\alpha| \\ &\leq |f_\alpha(\mathcal{P}_1) - \int_a^c f d\alpha| + |f_\alpha(\mathcal{P}_2) - \int_c^b f d\alpha| < \epsilon.\end{aligned}$$

Hence, the function f is McShane-Stieltjes integrable with respect to α on $[a, b]$ and $\int_a^b f d\alpha = \int_a^c f d\alpha + \int_c^b f d\alpha$. □ □

The next theorem shows the linearity of the McShane-Stieltjes integrals.

THEOREM 2.6. *Let α be an increasing function on $[a, b]$. Let f and g be McShane-Stieltjes integrable with respect to α on $[a, b]$. Then*

(1) *kf is McShane-Stieltjes integrable with respect to α on $[a, b]$ and $\int_a^b kf d\alpha = k \int_a^b f d\alpha$ for each $k \in R$;*
(2) *$f + g$ is McShane-Stieltjes integrable with respect to α on $[a, b]$ and $\int_a^b (f + g) d\alpha = \int_a^b f d\alpha + \int_a^b g d\alpha$.*

Proof. (1) Let f be McShane-Stieltjes integrable with respect to α on $[a, b]$. Case 1. $k = 0$. Of course, (1) is obvious. Case 2. $k \neq 0$. There exists a positive function δ on $[a, b]$ such that $|f_\alpha(\mathcal{P}) - \int_a^b f d\alpha| < \frac{\epsilon}{|k|}$ whenever $\mathcal{P}$ is a free tagged partition of $[a, b]$ that is subordinate to δ. Then

$$|(kf)_\alpha(\mathcal{P}) - k \int_a^b f d\alpha| = |k| \cdot |f_\alpha(\mathcal{P}) - \int_a^b f d\alpha| < \epsilon.$$

Hence, kf is McShane-Stieltjes integrable with respect to α on $[a, b]$ and $\int_a^b kf d\alpha = k \int_a^b f d\alpha$ for each $k \in R$.

(2) Let f and g be McShane-Stieltjes integrable with respect to α on $[a, b]$. There exists a positive function δ_1 on $[a, b]$ such that $|f_\alpha(\mathcal{P}_1) - \int_a^b f d\alpha| < \frac{\epsilon}{2}$ whenever $\mathcal{P}_1$ is a free tagged partition of $[a, b]$ that is subordinate to δ_1 and a positive function δ_2 such that $|g_\alpha(\mathcal{P}_2) - \int_a^b g d\alpha| < \frac{\epsilon}{2}$

whenever $\mathcal{P}_2$ is a free tagged partition of $[a, b]$ that is subordinate to δ_2. Define δ on $[a, b]$ by

$$\delta(x) = \min\{\delta_1(x), \delta_2(x)\}.$$

Let $\mathcal{P}$ be a free tagged partition of $[a, b]$ that is subordinate to δ. Then

$$\begin{aligned}
&|(f+g)_\alpha(\mathcal{P}) - (\int_a^b f d\alpha + \int_a^b g d\alpha)| \\
&\leq |f_\alpha(\mathcal{P}) - \int_a^b f d\alpha| + |g_\alpha(\mathcal{P}) - \int_a^b g d\alpha| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Hence, $f + g$ is McShane-Stieltjes integrable with respect to α on $[a, b]$ and $\int_a^b (f+g) d\alpha = \int_a^b f d\alpha + \int_a^b g d\alpha$. □ □

We will use the following notation : $f^+(x) = \max\{f(x), 0\}$, $f^-(x) = \max\{-f(x), 0\}$.

THEOREM 2.7. *Let α be an increasing function on $[a, b]$. Let f and g are McShane-Stieltjes integrable with respect to α on $[a, b]$. If $\alpha \in C^1[a, b]$, then*

(1) *if $f \leq g$ nearly everywhere on $[a, b]$, then*

$$\int_a^b f d\alpha \leq \int_a^b g d\alpha,$$

(2) *if $f = g$ nearly everywhere on $[a, b]$, then*

$$\int_a^b f d\alpha = \int_a^b g d\alpha.$$

Proof. (1) Suppose that $f \geq 0$ nearly everywhere on $[a, b]$. Since $f^- = 0$ nearly everywhere on $[a, b]$, it is McShane-Stieltjes integrable with respect to α on $[a, b]$ by Theorem 2.2 and $\int_a^b f^- d\alpha = 0$. By Theorem 2.6(2), the function $f^+ = f + f^-$ is McShane-Stieltjes integrable with respect to α on the $[a,b]$. Since $f^+ \geq 0$ on $[a,b]$, it is $\int_a^b f^+ d\alpha \geq 0$. Consequently, by Theorem 2.6(1) and (2),

$$\int_a^b f d\alpha = \int_a^b (f^+ - f^-) d\alpha = \int_a^b f^+ d\alpha - \int_a^b f^- d\alpha = \int_a^b f^+ d\alpha \geq 0$$

The general result now follows since $g - f \geq 0$ nearly everywhere on $[a,b]$ implies

$$\int_a^b g d\alpha - \int_a^b f d\alpha = \int_a^b (g-f) d\alpha \geq 0.$$

(2) If $f = g$ nearly everywhere on $[a,b]$, then $f - g = 0$ nearly everywhere on $[a,b]$. By Theorem 2.2, $f - g$ is McShane-Stieltjes integrable with respect to α on $[a,b]$ and $\int_a^b (f-g) d\alpha = 0$. By Theorem 2.6, the function $g = f + (g - f)$ is McShane-Stieltjes integrable on $[a,b]$ with respect to α and

$$\int_a^b g d\alpha = \int_a^b f d\alpha + \int_a^b (g-f) d\alpha = \int_a^b f d\alpha. \square$$

$\square$

Corollary 2.8. *Let α be an increasing function on $[a,b]$. Let f and g are McShane-Stieltjes integrable with respect to α on $[a,b]$. If $f(x) \leq g(x)$ for all $x \in [a,b]$, then $\int_a^b f d\alpha \leq \int_a^b g d\alpha$.*

Theorem 2.9. *If f is continuous on $[a,b]$ and α is increasing on $[a,b]$, there exists $c \in [a,b]$ such that $\int_a^b f d\alpha = f(c)[\alpha(b) - \alpha(a)]$.*

Proof. If $\alpha(b) = \alpha(a)$, then any value of c in $[a,b]$ gives the desired conclusion. Suppose that $\alpha(b) > \alpha(a)$. Then f attains its maximum M and minimum m on $[a,b]$. We have

$$m[\alpha(b) - \alpha(a)] \leq \int_a^b f d\alpha \leq M[\alpha(b) - \alpha(a)]$$

Therefore

$$m \leq \frac{\int_a^b f d\alpha}{\alpha(b) - \alpha(a)} \leq M.$$

By the intermediate-value theorem, there exists c in $[a, b]$ such that

$$f(c) = \frac{\int_a^b f d\alpha}{\alpha(b) - \alpha(a)}. \square$$

$\square$

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