Kangweon-Kyungki Math. Jour. **5** (1997), No. 2, pp. 177–183

ESTIMATIONS OF THE GENERALIZED REIDEMEISTER NUMBERS

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Abstract. Let $\sigma(X, x_0, G)$ be the fundamental group of a transformation group (X, G). Let $R(\varphi, \psi)$ be the generalized Reidemeister number for an endomorphism $(\varphi, \psi) : (X, G) \to (X, G)$. In this paper, our main results are as follows ; we prove some sufficient conditions for $R(\varphi, \psi)$ to be the cardinality of $Coker(1-(\varphi, \psi)_{\bar{\sigma}})$, where 1 is the identity isomorphism and $(\varphi, \psi)_{\bar{\sigma}}$ is the endomorphism of $\bar{\sigma}(X, x_0, G)$, the quotient group of $\sigma(X, x_0, G)$ by the commutator subgroup $C(\sigma(X, x_0, G))$, induced by (φ, ψ). In particular, we prove $R(\varphi, \psi) = |Coker(1 - (\varphi, \psi)_{\bar{\sigma}})|$, provided that (φ, ψ) is eventually commutative.

1. Introduction

F. Rhodes [5] initiated the study of the fundamental group $\sigma(X, x_0, G)$ of a transformation group (X, G), a group G of homeomorphisms of a space X, as a generalization of the fundamental group $\pi_1(X, x_0)$ of a topological space X. In [4], we defined the generalized Reidemeister number $R(\varphi, \psi)$ for an endomorphism $(\varphi, \psi) : (X, G) \to (X, G)$ of a transformation group (X, G) and investigated the algebraic estimations of $R(\varphi, \psi)$.

The purpose of this paper is to prove some sufficient conditions for the generalized Reidemeister number $R(\varphi, \psi)$ to be the number of elements of $Coker(1 - (\varphi, \psi)_{\bar{\sigma}})$, where 1 is the identity isomorphism and $(\varphi, \psi)_{\bar{\sigma}}$ is the endomorphism of $\bar{\sigma}(X, x_0, G)$, the quotient group of $\sigma(X, x_0, G)$ by the commutator subgroup $C(\sigma(X, x_0, G))$, induced

Received June 27, 1997.
1991 Mathematics Subject Classification: 55M20, 57M05.
Key words and phrases: $(\varphi, \psi)_\sigma$–equivalent, (algebraic) Reidemeister number, eventual commutativity.

by (φ,ψ). In particular, if $(\varphi,\psi):(X,G)\to(X,G)$ is eventually commutative, then

$$R(\varphi,\psi)=|Coker(1-(\varphi,\psi)_{\bar{\sigma}})|.$$

We always assume that the space X is a compact connected polyhedron. The reader may refer to [5] for more details on the fundamental group $\sigma(X,x_0,G)$ of a transformation group (X,G).

2. Definitions and lemmas

Let $(\varphi,\psi):(X,G)\to(X,G)$ be an endomorphism. Since $\varphi(gx)=(\psi g)(\varphi x)$ for every pair (x,g), if α is a path in X of order g with base-point x_0, then $\varphi\alpha$ is a path in X of order $\psi(g)$ with base-point $\varphi(x_0)$. Furthermore, if two path α and β of the same order g is homotopic, $\alpha\simeq\beta$, then $\varphi\alpha\simeq\varphi\beta$. Thus (φ,ψ) induces a homomorphism

$$(\varphi,\psi)_*:\sigma(X,x_0,G)\to\sigma(X,\varphi(x_0),G)$$

defined by $(\varphi,\psi)_*[\alpha;g]=[\varphi\alpha;\psi(g)]$.

If λ is a path from $\varphi(x_0)$ to x_0, then λ induces an isomorphism

$$\lambda_*:\sigma(X,\varphi(x_0),G)\to\sigma(X,x_0,G)$$

defined by $\lambda_*[\alpha;g]=[\lambda\rho+\alpha+g\lambda;g]$ for each $[\alpha;g]\in\sigma(X,\varphi(x_0),G)$, where $\rho(t)=1-t$. This isomorphism λ_* depends only on the homotopy class of λ.

For the composition

$$\sigma(X,x_0,G)\xrightarrow{(\varphi,\psi)_*}\sigma(X,\varphi(x_0),G)\xrightarrow{\lambda_*}\sigma(X,x_0,G),$$

we denote $\lambda_*(\varphi,\psi)_*=(\varphi,\psi)_\sigma$.

Definition 2.1. ([4]) Two elements $[\alpha;g_1]$, $[\beta;g_2]$ in $\sigma(X,x_0,G)$ are said to be $(\varphi,\psi)_\sigma$*–equivalent*, $[\alpha;g_1]\sim[\beta;g_2]$, if there exists $[\gamma;g]\in\sigma(X,x_0,G)$ such that

$$[\alpha;g_1]=[\gamma;g][\beta;g_2](\varphi,\psi)_\sigma([\gamma;g]^{-1}).$$

Note that the relation $\sim$ is an equivalence relation on $\sigma(X,x_0,G)$, and partitions $\sigma(X,x_0,G)$ into disjoint equivalence classes. Let $\sigma(X,x_0,G)'(\varphi,\psi)_\sigma$ be the set of equivalence classes of $\sigma(X,x_0,G)$ under $(\varphi,\psi)_\sigma$ –equivalence. The cardinality of $\sigma(X,x_0,G)'(\varphi,\psi)_\sigma$ called the *algebraic Reidemeister number* of $(\varphi,\psi)_\sigma$ and is denoted by $R_*(\varphi,\psi)_\sigma$.

DEFINITION 2.2. ([4]) For an endomorphism $(\varphi, \psi) : (X, G) \to (X, G)$, we define the *Reidemeister number* $R(\varphi, \psi)$ of (φ, ψ) to be the algebraic Reidemeister number of $(\varphi, \psi)_\sigma$, that is,

$$R(\varphi, \psi) = R_*(\varphi, \psi)_\sigma.$$

In Definition 2.2, note that $R(\varphi, \psi)$ is independent of the choice of the path λ from $\varphi(x_0)$ to x_0.

The following two lemmas will be used in obtaining our main results.

LEMMA 2.3. *Let $(\varphi, \psi)_\sigma : \sigma(X, x_0, G) \to \sigma(X, x_0, G)$ be a homomorphism and G be abelian. Then, for any $[\alpha; g_1], [\beta; g_2] \in \sigma(X, x_0, G)$,*

(1) $[\alpha; g_1][\beta; g_2] \sim [\beta; g_2](\varphi, \psi)_\sigma([\alpha; g_1])$.
(2) $[\alpha; g_1] \sim (\varphi, \psi)_\sigma([\alpha; g_1])$.

Proof. (1) It follows immediately from Definition 2.1, that is,

$$\begin{aligned}[\alpha; g_1][\beta; g_2] &\sim [\alpha; g_1]^{-1}([\alpha; g_1][\beta; g_2])(\varphi, \psi)_\sigma([\alpha; g_1]) \\ &\sim [\beta; g_2](\varphi, \psi)_\sigma([\alpha; g_1]).\end{aligned}$$

(2) By taking $[\beta; g_2] = [x_0'; e]$ in (1), where x_0' is the constant map $x_0' : I \to X$, we have

$$[\alpha; g_1] \sim (\varphi, \psi)_\sigma([\alpha; g_1]).\square$$

□

LEMMA 2.4. *Under the same assumptions as that in Lemma 2.3, if $[\alpha; g_1] \sim [\beta; g_2]$ implies $[\alpha; g_1][\gamma; g] \sim [\beta; g_2][\gamma; g]$ for any $[\gamma; g] \in \sigma(X, x_0, G)$, then*

$$[\alpha; g_1][\beta; g_2][\gamma; g_3] \sim [\beta; g_2][\alpha; g_1][\gamma; g_3]$$

for any $[\gamma; g_3] \in \sigma(X, x_0, G)$.

Proof. From (2) of Lemma 2.3, we have

$$\begin{aligned}[\alpha; g_1][\beta; g_2] &\sim (\varphi, \psi)_\sigma([\alpha; g_1][\beta; g_2]) \\ &\sim (\varphi, \psi)_\sigma([\alpha; g_1])(\varphi; \psi)_\sigma([\beta; g_2]),\end{aligned}$$

$$(\varphi, \psi)_\sigma([\alpha; g_1]) \sim [\alpha; g_1].$$

According to the hypothesis and the first result of Lemma 2.3,

$$\begin{aligned}(\varphi, \psi)_\sigma([\alpha; g_1])(\varphi, \psi)_\sigma([\beta; g_2]) &\sim [\alpha; g_1](\varphi, \psi)_\sigma([\beta; g_2]) \\ &\sim [\beta; g_2][\alpha; g_1].\end{aligned}$$

Again, from the hypothesis, we obtain

$$[\alpha; g_1][\beta; g_2][\gamma; g_3] \sim [\beta; g_2][\alpha; g_1][\gamma; g_3].$$

Hence we have the desired result. □ □

3. The estimations of $R(\varphi, \psi)$

Let $C(\sigma(X, x_0, G))$ be a commutator subgroup $\sigma(X, x_0, G)$ and let

$$\bar{\sigma}(X, x_0, G) = \sigma(X, x_0, G)/C(\sigma(X, x_0, G)).$$

Then $\theta_\sigma : \sigma(X, x_0, G) \to \bar{\sigma}(X, x_0, G)$ is a canonical homomorphism.

THEOREM 3.1. *([4]) If $(\varphi, \psi) : (X, G) \to (X, G)$ is an endomorphism and G is an abelian, then $R(\varphi, \psi) \geq |Coker(1 - (\varphi, \psi)_{\bar{\sigma}})|$, where 1 and $(\varphi, \psi)_{\bar{\sigma}}$ denote respectively the identity isomorphism and the endomorphism of $\bar{\sigma}(X, x_0, G)$ induced by (φ, ψ). Furthermore, if $\sigma(X, x_0, G)$ is abelian,*

$$R(\varphi, \psi) = |Coker(1 - (\varphi, \psi)_{\bar{\sigma}})|.$$

Now, in Theorem 3.2 and Theorem 3.4, we shall present some sufficient conditions in order that $R(\varphi, \psi)$ equals the number of elements of the set $Coker(1 - (\varphi, \psi)_{\bar{\sigma}})$.

Theorem 3.2. *Let G be an abelian. For any $[\alpha; g_1]$, $[\beta; g_2]$, $[\gamma; g_3] \in \sigma(X, x_0, G)$, if*

$$[\alpha; g_1][\beta; g_2][\gamma; g_3] \sim [\beta; g_2][\alpha; g_1][\gamma; g_3]$$

then

$$R(\varphi, \psi) = |Coker(1 - (\varphi, \psi)_{\bar{\sigma}})|.$$

Proof. Let $\eta_{\bar{\sigma}} : \bar{\sigma}(X, x_0, G) \to Coker(1 - (\varphi, \psi)_{\bar{\sigma}})$ be the natural projection. It is sufficient to prove that the epimorphism $\eta_{\bar{\sigma}}\theta_\sigma$ induces a monomorphism between the set of $(\varphi, \psi)_\sigma$–equivalent classes $\sigma(X, x_0, G)'(\varphi, \psi)_\sigma$ and $Coker(1 - (\varphi, \psi)_{\bar{\sigma}})$, that is, if $\eta_{\bar{\sigma}}\theta_\sigma([\alpha; g_1]) = \eta_{\bar{\sigma}}\theta_\sigma([\alpha'; g_1'])$, then $[\alpha; g_1] \sim [\alpha'; g_1']$.

(1) By the assumption of Theorem,

$$\begin{aligned}
&[\alpha; g_1][\beta; g_2]([\alpha; g_1]^{-1}[\beta; g_2]^{-1}[\gamma; g_3]) \\
&\sim [\beta; g_2][\alpha; g_1]([\alpha; g_1]^{-1}[\beta; g_2]^{-1}[\gamma; g_3]) \\
&= [\gamma; g_3].
\end{aligned}$$

(2) Since $\theta_\sigma([\gamma; g_3]) = \theta_\sigma([\gamma'; g_3'])$ means $[\gamma'; g_3'][\gamma; g_3]^{-1} \in ker\theta_\sigma$, $[\gamma'; g_3'][\gamma; g_3]^{-1}$ is a product of commutators. Applying (1) again, we have $[\gamma; g_3] \sim [\gamma'; g_3']$.

(3) Suppose that $\eta_{\bar{\sigma}}\theta_\sigma([\alpha; g_1]) = \eta_{\bar{\sigma}}\theta_\sigma([\alpha'; g_1'])$. From the natural projection $\eta_{\bar{\sigma}}$, there exists $\overline{[\mu; g]} \in \bar{\sigma}(X, x_0, G)$ and $[\gamma; g_3] \in \theta_\sigma^{-1}(\overline{[\mu; g]})$ such that

$$\begin{aligned}
\theta_\sigma([\alpha'; g_1']) - \theta_\sigma([\alpha; g_1]) &= (1 - (\varphi, \psi)_{\bar{\sigma}})(\overline{[\mu; g]}) \\
&= \overline{[\mu; g]} - (\varphi, \psi)_{\bar{\sigma}}(\overline{[\mu; g]}).
\end{aligned}$$

Consider the following commutative diagram ;

$$\begin{array}{ccc}
\sigma(X, x_0, G) & \xrightarrow{(\varphi,\psi)_\sigma} & \sigma(X, x_0, G) \\
{\scriptstyle \theta_\sigma}\downarrow & & {\scriptstyle \theta_\sigma}\downarrow \\
\bar{\sigma}(X, x_0, G) & \xrightarrow{(\varphi,\psi)_{\bar{\sigma}}} & \bar{\sigma}(X, x_0, G)
\end{array}$$

Since $(\varphi,\psi)_{\bar\sigma}(\overline{[\mu;g]}) = (\varphi,\psi)_{\bar\sigma}\theta_\sigma([\gamma;g_3]) = \theta_\sigma(\varphi,\psi)_\sigma([\gamma;g_3])$,

$$\begin{aligned}\theta_\sigma([\alpha';g_1']) &= \theta_\sigma([\alpha;g_1]) + \theta_\sigma([\gamma;g_3]) - \theta_\sigma((\varphi,\psi)_\sigma([\gamma;g_3]))\\ &= \theta_\sigma([\gamma;g_3][\alpha;g_1](\varphi,\psi)_\sigma([\gamma;g_3]^{-1})).\end{aligned}$$

From (2), we obtain

$$\begin{aligned}[\alpha';g_1'] &\sim [\gamma;g_3][\alpha;g_1](\varphi,\psi)_\sigma([\gamma;g_3]^{-1})\\ &\sim [\alpha;g_1].\end{aligned}$$

Therefore the proof of this theorem is complete. □ □

Let $(\varphi,\psi)_\sigma^k$ denote the k–th iterations of $(\varphi,\psi)_\sigma$.

Definition 3.3. Let G be an abelian. An endomorphism $(\varphi,\psi) : (X,G) \to (X,G)$ will be said to be *eventually commutative* if there exists a natural number k such that

$$(\varphi,\psi)_\sigma^k([\alpha;g_1][\beta;g_2]) = (\varphi,\psi)_\sigma^k([\beta;g_2][\alpha;g_1])$$

for each $[\alpha;g_1],[\beta;g_2] \in \sigma(X,x_0,G)$.

This means that $(\varphi,\psi)_\sigma^k(\sigma(X,x_0,G))$ is a commutative subgroup of $\sigma(X,x_0,G)$.

Theorem 3.4. *Let G be an abelian. If $(\varphi,\psi) : (X,G) \to (X,G)$ is eventually commutative, then $R(\varphi,\psi) = |Coker(1-(\varphi,\psi)_{\bar\sigma})|$.*

Proof. We want to show that the condition of Theorem 3.2 holds. By the assumption, there exists a natural number k such that

$$(\varphi,\psi)_\sigma^k([\alpha;g_1][\beta;g_2]) = (\varphi,\psi)_\sigma^k([\beta;g_2][\alpha;g_1]).$$

From (2) of Lemma 2.3,

$$\begin{aligned}[\alpha;g_1][\beta;g_2][\gamma;g_3] &\sim (\varphi,\psi)_\sigma^k([\alpha;g_1][\beta;g_2][\gamma;g_3])\\ &= (\varphi,\psi)_\sigma^k([\alpha;g_1][\beta;g_2])(\varphi,\psi)_\sigma^k([\gamma;g_3])\\ &= (\varphi,\psi)_\sigma^k([\beta;g_2][\alpha;g_1])(\varphi,\psi)_\sigma^k([\gamma;g_3])\\ &= (\varphi,\psi)_\sigma^k([\beta;g_2][\alpha;g_1][\gamma;g_3])\\ &\sim [\beta;g_2][\alpha;g_1][\gamma;g_3].\end{aligned}$$

This completes the proof. □ □

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