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A NOTE ON LIFTING TRANSFORMATION GROUPS

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Abstract. The purpose of this note is to compare two known results related to the lifting problem of an action of a topological group G on a G-space X to a coverring space of X.

1. Introduction

For a G-space X and a covering space $\tilde{X}_H$ of X associated with a subgroup H of $\pi_1(X, x_0)$, there exist some results related to the lifting problem of an action of G on X to an action of G on $\tilde{X}_H$. In this note, we show that the result due to M. A. Armstrong [1] is equivalent to a minor modification of the result due to F. Rhodes [2] under some restricted conditions. Also, we briefly refer to a role of the evaluation map with respect to the lifting problem.

We shall assume throughout this note that G is a locally path-connected topological group, that X is a path-connected, locally path-connected, and locally simply connected G-space and that $p : \tilde{X}_H \to X$ is a covering projection associated with a subgroup H of $\pi_1(X, x_0)$. Also, we use the following notations:

e: the identity element of G.

$\alpha * \beta$: the composition of two paths α and β.

$f \circ g$: the composition of two functions f and g.

i_X: the identity function on a set X.

$f_\#$: the homomorphism from $\pi_1(X, x_0)$ to $\pi_1(Y, f(x_0))$ induced by a map $f : X \to Y$.

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2. Preliminaries

For $g \in G$, let λ be a path from x_0 to gx_0. Define $g_* : \pi_1(X, x_0) \to \pi_1(X, x_0)$ by $g_*([\alpha]) = [\lambda * g\alpha * \lambda^{-1}]$ for $[\alpha] \in \pi_1(X, x_0)$. It is clear that for every normal subgroup H of $\pi_1(X, x_0)$, $g_*(H)$ is a normal subgroup which is independent of λ.

DEFINITION 2.1. ([2]) A normal subgroup H of $\pi_1(X, x_0)$ is said to be *G-invariant* if $g_*(H) = H$ for every $g \in G$.

DEFINITION 2.2. ([2]) Given $g \in G$, a *path α order g*, written by $(\alpha; g)$, with base point x_0 is a continuous function $\alpha : I \to X$ such that $\alpha(0) = x_0$ and $\alpha(1) = gx_0$.

LEMMA 2.3. *([2]) Let H be a subgroup of $\pi_1(X, x_0)$ and let $[\alpha; g]_H$ be the equivalence class of $(\alpha; g)$ under the equivalence relation*

$$(\alpha; g) \sim (\beta; h) \text{ iff } g = h \text{ and } [\alpha * \beta^{-1}] \in H.$$

If H is G-invariant normal, then the set $\sigma_H(X, x_0, G)$ of equivalence classes forms a group under the rule of composition

$$[\alpha; g]_H * [\beta; h]_H = [\alpha * g\beta; gh]_H.$$

LEMMA 2.4. *([2]) Let H be a subgroup of $\pi_1(X, x_0)$. If $\sigma_H(X, x_0, G)$ is a group, then we have a short exact sequence*

$$0 \to \pi_1(X, x_0)/H \xrightarrow{i} \sigma_H(X, x_0, G) \xrightarrow{j} G \to 0,$$

where $i([\alpha] * H) = [\alpha; e]_H$ and $j([\beta; g]_H) = g$.

From now on, j always denotes the homomorphism defined in Lemma 2.4.

In [2], a basis of open nbds is defined for the set $\sigma_H(X, x_0, G)$ as follows. Given $[\alpha; g]_H$ and open nbds U of gx_0 and V of e, define $W_X([\alpha; g]_H$,U,V) to be the set of classes $[\alpha * \beta; h]_H$ where $hg^{-1} \in V$ and β is a path in U from gx_0 to hx_0. Sets of the form of $W_X([\alpha; g]_H, U, V)$ constitude a basis for a topology on $\sigma_H(X, x_0, G)$.

F. Rhodes [2] showed that, if $\sigma_H(X, x_0, G)$ is a group, it is a topological group with the topology just defined.

Definition 2.5. ([2]) Let $\sigma_H(X, x_0, G)$ be a group. If there exists a continuous homomorphism $\phi : G \to \sigma_H(X, x_0, G)$ such that $j \circ \phi = i_G$, then the group $\sigma_H(X, x_0, G)$ is said to admit a *continuous split extension.*

Definition 2.6. We say that X admits a *family of H-preferred paths* at x_0 if it is possible to associate with every element g of G a path k_g from gx_0 to x_0 such that $[k_e] \in H$ and for every pair of elements g, h, the paths k_g, k_h and k_{gh} associated with g, h and gh satisfy $[gk_h * k_g * k_{gh}^{-1}] \in H$.

Definition 2.7. ([1]) Suppose that G also acts on a space Z, and that $f : Z \to X$ is a G-map which sends z_0 to x_0. If for every element g of G, loop α representing an element of H and path γ which joins z_0 to gz_0 in Z, $[(f\gamma) * g\alpha * (f\gamma^{-1})] \in H$, then H is said to be (f, G)-*invariant.*

3. Main Results

Lemma 3.1. *Let H be a normal subgroup of $\pi_1(X, x_0)$. If for every $g \in G$, $g_*(H) \subset H$, then $\sigma_H(X, x_0, G)$ is a group.*

Proof. Assume $[\alpha_1; g]_H = [\alpha_2; g]_H$ and $[\beta_1; h]_H = [\beta_2; h]_H$. Then $[\alpha_1 * \alpha_2^{-1}], [\beta_1 * \beta_2^{-1}] \in H$. Since $g^{-1}\alpha_2$ is a path from $g^{-1}x_0$ to x_0, $[g^{-1}\alpha_2^{-1} * g^{-1}(\beta_2 * \beta_1^{-1}) * g^{-1}\alpha_2] \in H$. From this, we obtain

$$\begin{aligned} &[(\alpha_1 * g\beta_1) * (\alpha_2 * g\beta_2)^{-1}] * [(\alpha_1 * \alpha_2^{-1}) * (\beta_1 * \beta_2^{-1})]^{-1} \\ =&[\alpha_1 * g(\beta_1 * \beta_2^{-1}) * g(g^{-1}\alpha_2^{-1} * g^{-1}(\beta_2 * \beta_1^{-1}) * g^{-1}\alpha_2) * \alpha_1^{-1}] \\ \in& g_*(H) \\ =&H. \end{aligned}$$

Thus $[(\alpha_1 * g\beta_1) * (\alpha_2 * g\beta_2)^{-1}] \in H$. This says that the binary operation is well defined. The other conditions for $\sigma_H(X, x_0, G)$ to be a group is obvious. □ □

LEMMA 3.2. *Let H be a subgroup of $\pi_1(X, x_0)$. If there exists a path connected space Z, and an action of G on Z, and a based G-map $f : (Z, z_0) \to (X, x_0)$ such that $f_\#(\pi_1(Z, z_0)) \subset H$, then X admits a family of H-preferred paths at x_0. Furthermore, if H is a normal subgroup of $\pi_1(X, x_0)$ such that $g_*(H) \subset H$ for all $g \in G$, then $\sigma_H(X, x_0, G)$ admits a continuous split extension.*

Proof. For each $g \in G$, choose a path γ_g in Z which joins gz_0 to z_0 and let $k_g = f\gamma_g$. By hypothesis, $[k_e] = [f\gamma_e] = f_\#([\gamma_e]) \in H$. If $g, h \in G$, then $g\gamma_h * \gamma_g * \gamma_{gh}^{-1}$ is a loop at z_0. Since $f_\#(\pi_1(X, x_0)) \subset H$, $[gk_h * k_g * k_{gh}^{-1}] \in H$. Thus $\{k_g | g \in G\}$ is a collection of H-preferred paths at x_0. Now, assume that $g_*(H) \subset H$ for all $g \in G$. By Lemma 3.1, $\sigma_H(X, x_0, G)$ is a group. Define $\phi : G \to \sigma_H(X, x_0, G)$ by $\phi(g) = [k_g^{-1}; g]_H$. Since $\{k_g | g \in G\}$ is a family of H-preferred paths,

$$\begin{aligned}\phi(g_1g_2) = [k_{g_1g_2}^{-1}; g_1g_2]_H &= [k_{g_1}^{-1} * g_1k_{g_2}^{-1}; g_1g_2]_H \\ &= [k_{g_1}^{-1}; g_1]_H * [k_{g_2}^{-1}; g_2]_H \\ &= \phi(g_1) * \phi(g_2).\end{aligned}$$

This shows that ϕ is a spliting homomorphism. Let $W_X([k_g^{-1}; g]_H, U, V)$ be a basis element containning $[k_g^{-1}; g]_H$. Choose an open nbd V_1 of e such that $V_1 \subset V$ and for any $h_1 \in V_1$, $h_1gx_0 \in U$. Also, choose an open nbd V_2 of e such that for all $h_2 \in V_2$, $h_2gz_0 \in f^{-1}(U)$. Let V' be the path component of $V_1 \cap V_2$ which contains e, let $g' \in V'g$ and let $c : I \to Vg$ be a path which joins g and g'. Then the map $g : I \to Z$, defined by $\gamma(s) = c(s)z_0$ is a path in $f^{-1}(U)$ which joins gz_0 to $g'z_0$, and hence $f\gamma$ is a path in U joining gx_0 to $g'x_0$. Since $[k_g^{-1} * (f\gamma) * k_{g'}] = f_\#([\gamma_g^{-1} * \gamma * \gamma_{g'}]) \in H$, we have $[k_{g'}^{-1}; g']_H = [k_g^{-1} * (f\gamma); g']_H \in W_X([k_g^{-1}; g]_H, U, V)$ and hence $\phi(V'g) \subset W_X([k_g^{-1}; g]_H, U, V)$. Consequently, ϕ is continuous. □ □

THEOREM 3.3. *Let H be a normal subgroup of $\pi_1(X, x_0)$ and let Z and f be the same as in Lemma 3.2. If*

(i) H is (f, G)-invariant and

(ii) $f_\#(\pi_1(Z, z_0)) \subset H$,

then $\sigma_H(X, x_0, G)$ is a group which admits a continuous split extension. Furthermore, $g_*(H) = H$ for every $g \in G$.

Proof. By Lemma 3.2, there exists a family $\{k_g | g \in G\}$ of H-preferred paths at x_0. Let $g \in G$ and $[\alpha] \in H$. Since for every $g \in G$, $g_*([\alpha]) = [k_g^{-1} * g\alpha * k_g] = [(f\gamma_g^{-1}) * g\alpha * (f\gamma_g)] \in H$ by (i), we have $g_*(H) \subset H$. By Lemma 3.1 and Lemma 3.2, $\sigma_H(X, x_0, G)$ is a group which admits a continuous split extension.

To show that $H \subset g_*(H)$, let $[\alpha] \in H$. Since $g\gamma_{g^{-1}} * \gamma_g$ is a loop in Z based at z_0, $[gk_{g^{-1}} * k_g] = f_\#([g\gamma_{g^{-1}} * \gamma_g]) \in H$ by (ii). Let $\beta = gk_{g^{-1}} * k_g$. Then

$$\begin{aligned}
[\alpha] &= [\beta^{-1} * (\beta * \alpha * \beta^{-1}) * \beta] \\
&= [k_g^{-1} * g(k_{g^{-1}}^{-1} * g^{-1}(\beta * \alpha * \beta^{-1}) * k_{g^{-1}}) * k_g] \\
&= g_*([k_{g^{-1}}^{-1} * g^{-1}(\beta * \alpha * \beta^{-1}) * k_{g^{-1}}]) \\
&= (g_* \circ g_*^{-1})([\beta * \alpha * \beta^{-1}]) \\
&\in g_*(H). \square
\end{aligned}$$

□

LEMMA 3.4. *Let $\sigma_H(X, x_0, G)$ be a group. Then X admits a family of H-preferred paths at x_0 if and only if the short exact sequence in Lemma 2.4 splits.*

Proof. ($\Rightarrow$) Define $\phi : G \to \sigma_H(X, x_0, G)$ by $\phi(g) = [\alpha_g^{-1}; g]_H$, where α_g is an H-preferred path associated with g. Clearly, $j \circ \phi = i_G$. Let $g, h \in G$. Since $[g\alpha_h * \alpha_g * \alpha_{gh}^{-1}] \in H$, we have $\phi(gh) = [\alpha_{gh}^{-1}; gh]_H = [\alpha_g^{-1} * g\alpha_h^{-1}; gh]_H = [\alpha_g^{-1}; g] * [\alpha_h^{-1}; h]_H = \phi(g) * \phi(h)$. Thus ϕ is a splitting homomorphism.

($\Leftarrow$) Let $\phi : G \to \sigma_H(X, x_0, G)$ be a splitting homomorphism. Then $\phi(e) = [c_{x_0}; e]_H$, where c_{x_0} is the constant path at x_0. For each $g \in G$, let $\phi(g) = [\alpha_g; g]_H$. Since $[\alpha_{gh}; gh]_H = \phi(gh) = \phi(g) * \phi(h) = [\alpha_g * g\alpha_h; gh]_H$, we have $[\alpha_g * g\alpha_h * \alpha_{gh}^{-1}] \in H$. Therefore, $\{\alpha_g^{-1} | g \in G\}$ is a collection of H-preferred paths at x_0. □ □

THEOREM 3.5. *Let H be a normal subgroup of $\pi_1(X, x_0)$. If $g_*(H) \subset H$ for every $g \in G$ and $\sigma_H(X, x_0, G)$ admits a continuous split extension, then the action of G lifts to an action of G on $\tilde{X}_H$.*

Proof. Define $\tilde{\mu} : \sigma_H(X, x_0, G) \times \tilde{X}_H \to \tilde{X}_H$ by $\tilde{\mu}([\alpha; g]_H, < \omega >) =< \alpha * g\omega >$ for $[\alpha; g]_H \in \sigma_H(X, x_0, G)$ and $< \omega > \in \tilde{X}_H$. Then $\tilde{\mu}$ is a well-defined action of $\sigma_H(X, x_0, G)$ on $\tilde{X}_H$.(see Proposition 2 of [2]) By hypothesis, there exists a continuous homomorphism $\phi : G \to \sigma_H(X, x_0, G)$ such that $j \circ \phi = i_G$. Let μ be the composition of

$$G \times \tilde{X}_H \overset{\phi \times i_{\tilde{X}_H}}{\longrightarrow} \sigma_H(X, x_0, G) \times \tilde{X}_H \overset{\tilde{\mu}}{\longrightarrow} \tilde{X}_H.$$

Clearly, μ covers the action of G on X. Let $\phi(g) = [\alpha_g; g]_H$ for $g \in G$. By Lemma 3.4, $\{\alpha_g^{-1} : g \in G\}$ is a family of H-preferred paths. Thus for $g_1, g_2 \in G$ and $< \omega > \in \tilde{X}_H$,

$$\begin{aligned}
\mu(g_1g_2, < \omega >) &=< \alpha_{g_1g_2} * (g_1g_2)\omega > \\
&=< \alpha_{g_1} * g_1\alpha_{g_2} * (g_1g_2)\omega > \\
&=< \alpha_{g_1} * g_1(\alpha_{g_2} * g_2\omega) > \\
&= \mu(g_1, < \alpha_{g_2} * g_2\omega >) \\
&= \mu(g_1, \mu(g_2, < \omega >)).
\end{aligned}$$

Since $\mu(e, < \omega >) =< \omega >$ for all $< \omega > \in \tilde{X}_H$, we conclude that μ is an action of G on $\tilde{X}_H$. □ □

Now, let $E : G \to X$ be the evaluation map define by $E(g) = gx_0$ for $g \subset G$.

LEMMA 3.6. *If N is a G-invariant subgroup of $\pi_1(G, e)$ such that $E_\#(N) \subset H$, then the map*

$$E_\#^R : \sigma_N(G, e, G) \to \sigma_H(X, x_0, G),$$

defined by $E_\#^R([\gamma; g]_N) = [E\gamma; g]_H$ for $[\gamma; g]_N \in \sigma_N(G, e, G)$, is a continuous homomorphism.

Proof. Clearly, $E_\#^R$ is a well-defined homomorphism. Now, let $[\gamma; g]_N \in \sigma_N(G, e, G)$ and let $W_X([w\gamma; g]_H, U, V)$ be an open neighborhood of $[E\gamma; g]_H$. Since E is continuous, there exists an open neighborhood U' of g such that $E(U') \subset U$. Let V' be an open neighborhood of e such that $V'g \subset U' \cap Vg$. Then for any $h \in V'g$ and any path γ' in U' from g to h, $h \in V_g$ and $E\gamma'$ is a path in U from gx_0 to hx_0. This means that

$$E^R_\#(W_G([\gamma; g]_N, U', V')) \subset W_X([E\gamma; g]_H, U, V).$$

Thus, $E^R_\#$ is continuous. □

□

LEMMA 3.7. *Let N be a G-invariant subgroup of $\pi_1(G, e)$ such that $E_\#(N) \subset H$. If $\sigma_N(G, e, G)$ admits a continuous split extension, then $\sigma_H(X, x_0, G)$ admits a continuous split extension.*

Proof. Consider the following commutative diagram

$$\begin{array}{ccc} \sigma_N(G, e, G) & \xrightarrow{j'} & G \\ E^R_\# \downarrow & & i_G \downarrow \\ \sigma_H(X, x_0, G) & \xrightarrow{j} & G \end{array}$$

where $j'([\gamma; g]_N) = g$ for $[\gamma; g]_N \in \sigma_G(G, e, G)$.

By hypothesis, there exists a continuous homomorphism $\phi' : G \to \sigma_N(G, e, G)$ such that $j' \circ \phi' = i_G$. Let $\phi = E^R_\# \circ \phi'$. By Lemma 3.6, ϕ is a continuous homomorphism. Since $j \circ \phi = j \circ (E^R_\# \circ \phi') = j' \circ \phi' = i_G$, $\sigma_H(X, x_0, G)$ admits a continuous split extension. □ □

LEMMA 3.8. *If $\pi_1(G, e) = N$, then $\sigma_N(G, e, G)$ admits a continuous split extension.*

Proof. By hypothesis, $j' : \sigma_N(G, e, G) \to g$ is an isomorphism. Let $\phi' = (j')^{-1}$. For $g \in G$, let $\phi'(g) = [\alpha_g; g]_H$ and let $W([\alpha_g; g]_H, U, V)$ be an open nbd of $[\alpha_g; g]_H$. Without loss of generality, we may assume that U is path connected. For $h \in Vg$, choose a path γ in U from gx_0 to hx_0. Since ϕ' is an isomorphism, $[\alpha_h; h]_H = [\alpha_g * \gamma; h]_H \in W([\alpha_g; g]_H, U, V)$, and hence $\phi'(Vg) \subset W([\alpha_g; g]_H, U, V)$. This implies that ϕ' is continuous. □ □

COROLLARY 3.9. *Let H be a G-invariant normal subgroup of $\pi_1(X, x_0)$. If $E_\#(\pi_1(G, e)) \subset H$, then the action of G on X lifts to an action of G on $\tilde{X}_H$.*

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