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UNIQUENESS OF SOLUTIONS FOR A DEGENERATE PARABOLIC EQUATION WITH ABSORPTION

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Abstract. We estimate the interior Lipschitz norm and maximum of the solution for degenerate parabolic equations with absorption. Also obtain the growth rate of the solution u in terms of time t. From this we show the uniqueness of solution with respect to the initial trace.

1. Introduction

We consider the Cauchy problem of a degenerate parabolic equation with absorption :

$$u_t = \operatorname{div}(|\nabla u|^{p-2}\nabla u) - bu^q \quad \text{in} \quad \mathbb{R}^n \times (0,\infty). \tag{1.1}$$

Here $p > 2$, $b \in [0,1]$ and $q \in (0,1)$ are given constants. A measurable function $u(x,t)$ defined in $\mathbb{R}^n \times (0,\infty)$ is a weak solution of (1.1) if for every bounded open set $\Omega \subset \mathbb{R}^n$

$$u \in C(0,T : L^1(\Omega)) \cap L^p(0,T : W^{1,p}(\Omega))$$

and satisfies

$$\begin{aligned} \int_\Omega u(x,t_2)\eta(x,t_2)\,dx + \int_{t_1}^{t_2}\int_\Omega -u\eta_t + |\nabla u|^{p-2}\nabla u\nabla\eta\,dxdt \\ = \int_\Omega u(x,t_1)\eta(x,t_1)\,dx - \int_{t_1}^{t_2}\int_\Omega bu^q\eta\,dxdt \end{aligned} \tag{1.2}$$

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for each bounded interval $[t_1, t_2] \subset (0, \infty)$ and all test functions η as

$$\eta \in W^{1,\infty}(0, T; L^\infty(\Omega)) \cap L^\infty(0, T; W_0^{1,\infty}(\Omega)).$$

A Radon measure μ on $\mathbb{R}^n$ is called the initial trace of u if μ satisfies

$$\lim_{t\to 0} \int_{R^n} u(x,t)\eta(x)dx = \int_{R^n} \eta\, d\mu$$

for all continuous functions η in $\mathbb{R}^n$ with compact support. And we say $u(x,t)$ is a weak solution to (1.1) with the initial trace μ.

When $p = 2$, Brezis and Friedmann[1] showed that (1.1) has a fundamental solution if and only if $q \in (0, (n+2)/n)$. And Brezis, Peletier and Terman[2] proved the existence of a very singular solution such that

$$\lim_{t\to 0} \int_{\mathbb{R}^n} u(x,t)dx = \infty$$

when $q \in (1, (n+2)/n)$. For the porous medium equation with absorption

$$u_t = \triangle u^m - u^p, \quad m > 1 \quad \text{in} \quad \mathbb{R}^n \times (0, \infty), \tag{1.3}$$

Kamin, Peletier and Vazquez [9], [10] considered that the problem(1.3) with $p > 1$. They proved that (1.3) has a fundamental solution if $p \in (1, m + 2/n)$ and that if $p \in (m, m + 2/n)$ then there exists a very singular solution, also in case of $p \geq m + 2/n$ there is no fundamental solution. And Cho [3] prove the existence of fundamental solution and existence of initial trace of weak solution when $p \in (0, 1)$.

The evolutionary p-Laplace equation

$$u_t - \mathrm{div}(|\nabla u|^{p-2}\nabla u) = 0 \quad \text{in} \quad \mathbb{R}^n \times (0, T) \quad p > 2 \tag{1.4}$$

has been studied by many authors [4], [6], [8],$\cdots$. When the space dimension is one, Kalakshnikov proved the existence of a unique solution of (1.4) for some small T with the condition of initial datum

$$|u_0(x)| \leq c(1 + |x|^2)^{\frac{p}{2(p-2)}} \quad \text{for} \quad x \in \mathbb{R} \quad \text{and for some} \quad c > 0.$$

For higher dimension, DiBenedetto and Herrero [6] showed the existence of initial trace and a weak solution to (1.4) in $\mathbb{R}^n \times (0,T)$, where T is

$$T(\mu) = c_0[\lim_{r\to\infty} \|\mu\|_r]^{-(p-2)} \quad \text{if} \quad \|\mu\|_r > 0$$

and $T(\mu) = \infty$ if $\|\mu\|_r = 0$. For the p-Laplace equation with bounded measurable coefficient, Choe and Lee [5] establish the Harnack type inequality and existence of initial trace.

For the p-Laplace equation with absorption,

$$u_t = \operatorname{div}(|\nabla u|^{p-2}\nabla u) - u^q \quad \text{in} \quad \mathbb{R}^n \times (0,T] \quad p > 2.$$

Peletier and Wang[12] showed that when $p-1 < q < p-1+p/n$ there exists a very singular solution.

For $0 < q < 1$, Lee [11] showed the existence of fundamental solution. And also proved the existence of unique initial traces μ of weak solutions of (1.1) satisfying

$$\sup_{R \geq T_0^{\frac{p-q-1}{p(1-q)}}} R^{-\frac{\kappa}{p-2}} \int_{B_R} d\mu < c(u(0,T)).$$

In section 2 we estimate interior Lipschitz norm in terms of L^p norm of u by Moser type iterations. After this, the maximum of u can be estimated by L^1 norm of u. From this we can obtain the growth rate of weak solution u in terms of t. Once we know the growth rate of u, we can show the following estimate

$$\int_0^\tau \int_{R^n} |\nabla u|^{p-1} dxdt \leq c\tau^{\frac{1}{\kappa}},$$

where $\kappa = n(p-2)+p$. This estimate is useful in showing the uniqueness of the solution.

In section 3, we prove that solutions are uniquely determined by their initial trace. On the process of proof we need higher integrability of u. Once this is shown, we can prove the uniqueness by use of the Granwall type inequality.

The following symbols are used;

$$\begin{aligned}
&B_R(x_0) = \{x : |x - x_0| < R\},\\
&Q_R(x_0, t_0) = B_R(x_0) \times (t_0 - R^p, t_0),\\
&S_R(x_0, t_0) = B_R(x_0) \times (t_0 - R^p, t_0 + R^p).
\end{aligned}$$

If there is no confusion, we drop out (x_0, t_0) in various expressions.

2. Interior estimate

In this section we prove various a priori estimates which are useful in studying pointwise behavior of u. First we prove a local maximum estimate by Moser iteration method.

LEMMA 2.1. *Suppose u is a nonnegative weak solution of (1.1) in Q_R, then there exist constants c_1 and c_2 depending on p, n, and R such that*

$$\sup_{Q_{\frac{R}{2}}} u \le c_1 \left[\iint_{Q_R} u^p dxdt \right]^{\frac{1}{2}} + c_2.$$

Proof. Let $\rho < R$. We take $u^{\alpha+1}\eta^p$ as a test function to (1.1), where η is a standard cutoff function such that $0 \le \eta \le 1$, $\eta = 0$ on $\partial_p Q_R$, and $\eta(x,t) = 1$ for all $(x,t) \in Q_\rho$, $|\nabla \eta| \le \frac{c}{(R-\rho)}$ and $|\eta_t| \le \frac{c}{(R-\rho)^p}$ for some constant c. Then we have

$$\iint u_t(u^{\alpha+1}\eta^p) + |\nabla u|^{p-2}\nabla u \nabla(u^{\alpha+1}\eta^p) + bu^q(u^{\alpha+1}\eta^p) dxdt = 0$$

The last term of the integrand is nonnegative so we have

$$\begin{aligned}
\frac{1}{(\alpha+2)} &\iint (u^{\alpha+2}\eta^p)_t \, dx\, dt - \frac{p}{(\alpha+2)} \iint u^{\alpha+2}\eta^{p-1}|\eta_t| dx\, dt\\
&+ (\alpha+1) \iint |\nabla u|^p u^\alpha \eta^p dx\, dt\\
&\le p \iint |\nabla u|^{p-1} u^{\alpha+1} \eta^{p-1} |\nabla \eta| dxdt
\end{aligned}$$

Hence from Young's inequality we have

$$
\begin{aligned}
&\sup_t \int u^{\alpha+2}\eta^p dx + (\alpha+1)\iint u^{\alpha}|\nabla u|^p \eta^p dxdt \\
&\le c(\alpha)\iint u^{\alpha+2}\eta^{p-1}|\eta_t| + u^{\alpha+p}\eta^{p-1}|\nabla\eta|^p dxdt \qquad (2.1)\\
&\le \frac{c(\alpha)}{(R-\rho)^p}\iint_{Q_R} u^{\alpha+2} + u^{\alpha+p} dxdt.
\end{aligned}
$$

Similarly we also have

$$
\begin{aligned}
&\sup_t \int u^{\alpha+2}\eta^p dx + c\iint |\nabla (u^{\frac{\alpha+p}{p}})\eta|^p dx\, dt \qquad (2.2)\\
&\quad \le \frac{c(\alpha)}{(R-\rho)^p}\iint \left[u^{\alpha+2} + u^{\alpha+p}\right] dxdt.
\end{aligned}
$$

From Hölder inequality and Sobolev inequality we have

$$
\begin{aligned}
&\iint_{Q_\rho} u^{\frac{p(\alpha+2)}{n}+(\alpha+p)} dxdt \\
&\le \int \left[\int u^{\alpha+2}\eta^p dx\right]^{\frac{p}{n}} \left[\int u^{\frac{\alpha+p}{p}\frac{np}{n-p}}\eta^{\frac{np}{n-p}} dx\right]^{\frac{n-p}{n}} dt \\
&\le \sup_t \left[\int u^{\alpha+2}\eta^p dx\right]^{\frac{p}{n}} \iint |\nabla (u^{\frac{\alpha+p}{p}})\eta|^p dx\, dt \\
&\le c\left[\iint u^{\alpha+2}\eta^{p-1}|\eta_t| + u^{\alpha+p}|\nabla\eta|^p dxdt\right]^{1+\frac{p}{n}} \qquad (2.3)\\
&\le \left[\frac{c}{(R-\rho)^p}\iint_{Q_R} u^{\alpha+p} dxdt + 1\right]^{1+\frac{p}{n}}
\end{aligned}
$$

for some c. We use Moser iteration method. Let $\alpha_{i+1}+p = (\alpha_i+2)\frac{p}{n} + (\alpha_i+p)$ and $\alpha_0 = 0$. Setting $\gamma = 1+\frac{p}{n}$, we can write $\alpha_i = 2(\gamma^i - 1)$. Define $R_i = \frac{1}{2}R(1+2^{-i})$, and take $\alpha = \alpha_i, \rho = R_{i+1}$ and $R = R_i$ in (2.3). Hence if we define $\Psi_i = \iint_{Q_{R_i}} u^{\alpha_i+p} dxdt$, we get from (2.3)

$$
\Psi_{i+1} \le c[\Psi_i + 1]^{\gamma}. \qquad (2.4)
$$

Iterating (2.4) we obtain

$$\left[\iint_{Q_{R_i}} u^{\alpha_i+p} dxdt\right]^{\frac{1}{\alpha_i+p}} \leq \left[c_1 \iint_{Q_R} u^p dx, dt\right]^{\frac{\gamma^i}{\alpha_i+p}} + c_2^{\frac{\gamma^i}{\alpha^i+p}}$$

for some c_1 and c_2. Letting $i \to \infty$ we get the result. □ □

Now we estimate $\iint_{Q_R} u^p dxdt$ in terms of $\iint_{Q_{2R}} udxdt$.

LEMMA 2.2. *Suppose u is a nonnegative weak solution of (1.1) in S_R. Then there are constants σ and c depending on n and p such that*

$$\sup_{S_{\frac{R_0}{2}}} u \leq c \left[\sup_t \int_{|x|<2R_0} u(x,t)dx + 1\right]^\sigma.$$

Proof. Let $\rho < R$. We take $u_1^{\alpha+1}\eta^p$ as a test function to (1.1), where $u_1 = u+1$ and η is a standard cutoff function such that $0 \leq \eta \leq 1$, $\eta = 0$ on $\partial_p S_R$, $\eta(x,t) = 1$ for all $(x,t) \in S_\rho$, $|\nabla\eta| \leq \frac{c}{(R-\rho)}$ and $|\eta_t| \leq \frac{c}{(R-\rho)^p}$ for some positive constant c. Then we have

$$\iint u_t(u_1^{\alpha+1}\eta^p) + |\nabla u|^{p-2}\nabla u \nabla(u_1^{\alpha+1}\eta^p) + bu^q(u_1^{\alpha+1}\eta^p)dxdt = 0$$

So we have

$$\begin{aligned}
&\sup_t \int u_1^{\alpha+2}\eta^p dx + (\alpha+1)\iint u_1^\alpha |\nabla u|^p \eta^p dxdt \\
&\leq \frac{c(\alpha)}{(R-\rho)^p}\iint u_1^{\alpha+2} + p\iint u_1^{\alpha+p}|\nabla\eta|^p dxdt \qquad (2.5)\\
&\leq \frac{c(\alpha)}{(R-\rho)^p}\iint u_1^{\alpha+2} + u_1^{\alpha+p} dxdt.
\end{aligned}$$

Similarly we also have

$$\begin{aligned}
&\sup_t \int u_1^{\alpha+2}\eta^p dx + c\iint |\nabla(u_1^{\frac{\alpha+p}{p}})\eta|^p dx\,dt \qquad (2.6)\\
&\qquad \leq \frac{c(\alpha)}{(R-\rho)^p}\iint \left[u_1^{\alpha+2} + u_1^{\alpha+p}\right] dxdt.
\end{aligned}$$

From Hölder inequality and Sobolev inequality we have

$$
\begin{aligned}
&\iint_{S_\rho} u_1^{(\alpha+2)\frac{p}{n}+(\alpha+p)}\eta dxdt \\
&\qquad \leq \int \left[\int u_1^{\alpha+2}\eta^p dx\right]^{\frac{p}{n}} \left[\int u_1^{\frac{\alpha+p}{p}\frac{np}{n-p}}\eta^{\frac{np}{n-p}}dx\right]^{\frac{n-p}{n}} dt \\
&\qquad \leq \sup_t \left[\int u_1^{\alpha+2}\eta^p dx\right]^{\frac{p}{n}} \iint |\nabla(u_1^{\frac{\alpha+p}{p}})\eta|^p dx\, dt \\
&\qquad \leq \left[\frac{c}{(R-\rho)^p}\iint_{S_R} u_1^{\alpha+p}dxdt + 1\right]^{1+\frac{p}{n}}
\end{aligned}
$$

On the other hand from Hölder inequality we have

(2.7)
$$
\begin{aligned}
&\iint u_1^{p/n+(\alpha+p)}\eta^{p(1+p/n)}\, dx\, dt \\
&\leq \left[\sup_t \int u_1\eta^p\, dx\right]^{p/n} \int \left[\int u_1^{(\alpha+p)n/(n-p)}\eta^{np/(n-p)}\, dx\right]^{(n-p)/n} dt.
\end{aligned}
$$

From (2.6) and Sobolev embedding theorem we obtain

$$
\begin{aligned}
\int \left[\int u_1^{\frac{\alpha+p}{p}\frac{np}{n-p}}\eta^{\frac{np}{n-p}}dx\right]^{\frac{n-p}{np}\cdot p} dt &\leq \iint |\nabla(u_1^{\frac{\alpha+p}{p}}\eta)|^p dxdt \\
\leq c\iint_{S_R}({u_1}^{\alpha+2}+{u_1}^{\alpha+p})dxdt &\leq c\iint_{S_R} u^{\alpha+p}+1dxdt.
\end{aligned}
$$

Then we can write (2.7) as

$$
(2.8)\quad \iint_{S_\rho} {u_1}^{\frac{p}{n}+(\alpha+p)}dxdt \leq c\left[\sup_t \int_{|x|<R} u_1 dx\right]^{\frac{p}{n}} \iint_{S_R} {u_1}^{\alpha+p}dxdt.
$$

Define $I = \sup_t \int_{|x|<2R_0} dx$ and let $\frac{p}{n}+\alpha_i+p = \alpha_{i+1}+p$ with $\alpha_0+p = 1$ then $\alpha_i = \alpha_0 + \frac{p}{n}i$.

Define $R_i = R_0(1+2^{-i})$ and $\rho = R_{i+1}$, $R = R_i$. Hence iterating (2.8) we obtain

$$\iint_{S_{R_{i+1}}} u_1^{\alpha_{i+1}+p} dx\, dt \le c I^{\frac{p}{n}} \iint_{S_{R_i}} u_1^{(\alpha_i+p)} dxdt$$

and
(2.9)

$$\iint_{S_{R_i}} u_1^{\alpha_i+p} dxdt \le c I^{\sigma} \left[\iint_{S_{R_0}} u_1 dxdt \right] \le c \left[\sup_t \int_{|x|<2R_0} dx \right]^{\sigma+1}$$

for some σ depending only on n, p. Therefore combining Lemma 2.1 and (2.9) we prove the Lemma. □ □

Now we improve Lemma 2.2.

THEOREM 2.3. *Let u be a nonnegative weak solution of (1.1) in S_{2R}. Then there are constants c, γ and σ depending on p, n such that*

$$\sup_{S_{\frac{R_0}{2}}} u \le c\left[I^{\sigma} + I^{\gamma}\right], \tag{2.10}$$

where $I = \sup_t \int_{|x|<2R_0} u(x,t)dx$.

Proof. Let ε_0 be a some fixed constant. If $I \ge \varepsilon_0$, then Lemma 2.2 implies (2.10) with a constant c depending on ε_0. Now we assume $I < \varepsilon_0$ then there is $c_0(R_0)$ such that $0 \le u \le c_0$ in Q_{R_0}. Since $\sup|u| \le c_0$ from (2.1) we can deduce that

$$\sup_t \int u^{\alpha+2} \eta^p dx + c \iint |\nabla (u^{\frac{\alpha+p}{p}})\eta|^p dxdt \le c \iint u^{\alpha} \tag{2.11}$$

for some constant c depending on ε_0, $||\nabla\eta||_\infty, ||\eta_t||_\infty$ and c_0 . Hence iterating (2.11) with similar methods as Lemma 2.1 and 2.2 we obtain

$$\sup_{Q_{\frac{R_0}{2}}} u \le c \left[\iint_{S_{2R_0}} udxdt \right]^{\delta_0} \le c I^{\delta_0}$$

for some c and δ_0 depending on ε_0, n and p. $\square$ $\square$

We denote by S the class of all nonnegative weak solutions of (1.1) in $\mathbb{R}^n \times (0,\infty)$ and we define a subclass $P(N)$ by

$$P(N) = \{u \in S : \sup_t \int_{R^n} u(x,t)dx \leq N\}$$

LEMMA 2.4. *Let $u \in P(N)$, then there is a constant $c(N)$ depending on p, n, q and N such that*

$$u(x,t) \leq c(N)t^{\frac{-n}{n(p-2)+p}} \quad \textit{for} \quad 0 < t < 1.$$

Proof. We define $v(\xi,\tau)$ as

$$v(\xi,\tau) = \frac{1}{\gamma}u(x+R\xi, t\tau), \quad \text{with} \quad R \geq t^{\frac{p-q-1}{p(1-q)}} \tag{2.12}$$

where γ is defined by $\gamma^{(p-2)} = \frac{R^p}{t}$. Then $v(\xi,\tau)$ is a solution to

$$\frac{\gamma}{t}v_\tau(\xi,\tau) = \frac{\gamma^{p-1}}{R^p}\text{div}(|\nabla v|^{p-2}\nabla v) - b\gamma^q v^q$$

which is

$$v_\tau(\xi,\tau) == \text{div}(|\nabla v|^{p-2}\nabla v) - bt^{\frac{p-q-1}{p-2}}R^{\frac{p(q-1)}{p-2}}v^q.$$

We note that $R \geq t^{\frac{p-q-1}{p(1-q)}}$, then $bt^{\frac{p-q-1}{p-2}}R^{\frac{p(q-1)}{p-2}} \in [0,1]$. We can choose R as $\gamma R^n = \frac{N}{M}$ for some constant M such that $R = c\frac{N}{M}^{\frac{p-2}{\kappa}}t^{\frac{1}{\kappa}}$ greater than or equal to $t^{\frac{p-q-1}{p(1-q)}}$ then

$$\int_{\mathbb{R}^n} v(\xi,\tau)\,d\xi = \frac{1}{\gamma}\int_{\mathbb{R}^n} u(x+R\xi,t\tau)\,d\xi = \frac{1}{\gamma R^n}\int_{\mathbb{R}^n} u(y,t\tau)dy \leq M$$

and $v \in P(M)$. Therefore from Theorem 2.3, we get

$$u(x,t) = \gamma v(0,1) = \left(\frac{R^p}{t}\right)^{\frac{1}{p-2}} v(0,1) \leq c\left(\frac{R^p}{t}\right)^{\frac{1}{p-2}} = c(N)t^{\frac{-n}{n(p-2)+p}}$$

and the proof is completed. $\square$ $\square$

LEMMA 2.5. *Let $u \in P(N)$, then there exists some positive constant c depending on σ, p and n such that*

$$\int_0^\tau \int_{B_R} |\nabla u|^{p-1}\,dxdt \le c(N)\tau^{\frac{1}{\kappa}} \tag{2.13}$$

for all $\tau \in (0,1)$, where $\kappa = n(p-2)+p$

Proof. By Hölder's inequality we get

$$\begin{aligned}
&\int_0^\tau \int_{B_R} |\nabla u|^{p-1}\,dxdt \\
&= \int_0^\tau \int_{B_R} t^\delta u^{-\varepsilon}|\nabla u|^{p-1} t^{-\delta}u^\varepsilon\,dxdt \\
&\le \left[\int_0^\tau \int_{B_R} (t^\delta u^{-\varepsilon})^{\frac{p}{p-1}}|\nabla u|^p\,dxdt\right]^{\frac{p-1}{p}} \left[\int_0^\tau \int_{B_R} (t^{-\delta}u^\varepsilon)^p\,dxdt\right]^{\frac{1}{p}}.
\end{aligned}$$

Let $A = \int_0^\tau \int_{B_R} \left(t^\delta u^{-\varepsilon}\right)^{\frac{p}{p-1}} |\nabla u|^p\,dxdt$ and $B = \int_0^\tau \int_{B_R} \left(t^{-\delta}u^\varepsilon\right)^p\,dxdt$ From Lemma 2.4, B can be estimated as

$$\begin{aligned}
B &\le \int_0^\tau t^{-\delta p}\|u^{\varepsilon p-1}\|_\infty \int_{B_R} u\,dxdt \le c(N)\int_0^\tau t^{-p\delta}t^{(\varepsilon p-1)\frac{-n}{\kappa}}\,dt \\
&= c\int_0^\tau t^{-p\delta-\frac{n}{\kappa}(\varepsilon p-1)}\,dt.
\end{aligned}$$

Hence if we choose δ and ε satisfying $p\delta + \frac{n}{\kappa}(\varepsilon p-1) < 1$ and $\varepsilon > \frac{1}{p}$, then $B \le c\tau^{1-p\delta-\frac{n}{\kappa}(\varepsilon p-1)}$ for some c depending only on ρ, n and p.

To estimate A, take $t^{\frac{\delta p}{p-1}}u^{1-\frac{\varepsilon p}{p-1}}\phi^2$ as a test function to (1.1), where ϕ is a piecewise smooth cutoff function in B_{R+1} with $|\nabla\phi| \le c$. Here we assume $1-\frac{\varepsilon p}{p-1} \ge 0$. From the definition of A and Young's inequality we obtain $A \le c\tau^{-\frac{n}{\kappa}(1-\frac{\varepsilon p}{p-1})+\frac{\delta p}{p-1}}$. For details, see [5,Lemma 3.2]. So with a suitable choice of δ and ε we conclude that

$$\begin{aligned}
\int_0^\tau \int_{B_R} |\nabla u|^{p-1}\,dxdt &\le \left[c\tau^{\frac{\delta p}{p-1}-\frac{n}{\kappa}(1-\frac{\varepsilon p}{p-1})}\right]^{\frac{p-1}{p}} \left[c\tau^{1-(\delta p+\frac{n}{\kappa}(\varepsilon p-1))}\right]^{\frac{1}{p}} \\
&= c\tau^{\frac{-np+2n+\kappa}{\kappa p}} = c\tau^{\frac{1}{\kappa}},
\end{aligned}$$

where c depends on n and p. Thus we get

$$\int_0^\tau \int_{B_R} |\nabla u|^{p-1}\, dxdt \le c(N)\tau^{\frac{1}{\kappa}}$$

for some c. $\square$ $\square$

LEMMA 2.6. *Let $u \in P(N)$, then there exists some positive constant c depending on p, n, N and R such that*

$$\int_0^\tau \int_{B_R} u^q\, dxdt \le \begin{cases} c\tau^{1-nq/k} & \textit{if} \quad q < \dfrac{\kappa}{n} \\ c\tau^{\frac{2}{3}} & \textit{if} \quad q = \dfrac{\kappa}{n} \\ c\tau^{1-\frac{nq}{2nq-k}} & \textit{if} \quad q > \dfrac{\kappa}{n} \end{cases}$$

for all $\tau \in (0,1)$, where $\kappa = n(p-2)+p$.

Proof. First, we assume $q < \frac{k}{n}$, then by Lemma 2.4, we have

$$\begin{aligned} \int_0^\tau \int_{B_R} u^q\, dxdt &\le R^n \int_0^\tau \|u\|_\infty^q dt \\ &\le c(N,R) \int_0^\tau t^{-\frac{n}{k}q} dt = c(N,R)\tau^{1-\frac{n}{k}q}. \end{aligned}$$

We assume $q = \frac{k}{n}$ and set $A = \frac{1}{2}$, then by Hölder inequality and Lemma 2.4, we have

$$\begin{aligned} \int_0^\tau \int_{B_R} u^{\frac{k}{n}}\, dxdt &= R^{n(1-\frac{k}{n(1+A)})} \int_0^\tau \left(\int_{B_R} u^{1+A}\, dx \right)^{\frac{k}{n(1+A)}} dt \\ &\le c(N,R) \int_0^\tau \|u\|_\infty^{-\frac{kA}{n(1+A)}}\, dt \\ &\le c(N,R) \int_0^\tau t^{-\frac{A}{1+A}}\, dt = c(N,R)\tau^{\frac{2}{3}}. \end{aligned}$$

We now assume $q > \frac{k}{n}$ and set $A = \frac{1}{2}(\frac{1}{q} + \frac{n}{nq-k})$ by the direct calculation we can show that $qA > 1$. From Hölder inequality, we have

$$\begin{aligned}\int_0^\tau \int_{B_R} u^q \, dxdt &= R^{n(1-\frac{1}{A})} \int_0^\tau (\int_{B_R} u^{qA} dx)^{\frac{1}{A}} dt \\ &\leq c(N,R) \int_0^\tau \|u\|_\infty^{q-\frac{1}{A}} dt \\ &\leq c(N,R) \int_0^\tau t^{-\frac{n}{k}(q-\frac{1}{A})} dt = c(N,R)\tau^{1-\frac{nq}{2nq-k}}\end{aligned}$$

so the proof is completed. □ □

3. Uniqueness of solution

In this section we will show that solutions are uniquely determined by their initial traces. The existence of initial traces for each solution u of (1.1) was showed by Lee[11].

Lemma 3.1.. *Suppose that u and v are two weak solutions of (1.1) in $\mathbb{R}^n \times (0,T)$ for some $0 < T < \infty$. If*

$$\sup_{t\in(0,T)} [\|u(t)\|_1 + \|v(t)\|_1] < \infty \quad \text{and} \quad \lim_{t\to 0} [u(\cdot,t) - v(\cdot,t)] = 0$$

in $L^1_{loc}(\mathbb{R}^n)$ then

$$\lim_{t\to 0} [u(\cdot,t) - v(\cdot,t)] = 0 \quad \text{in} \quad L^{1+\varepsilon}_{loc} \quad \text{for all} \quad 0 < \varepsilon < \frac{1}{n}.$$

Proof. Fix $\varepsilon < \frac{1}{n}$ and let $w = u - v$. It suffices to show that for each $R \geq 1$

$$\lim_{t\to 0} \int_{B_R} |w(x,t)|^{1+\varepsilon} dx = 0.$$

From Lemma 2.4, we know that

$$\sup_{x\in B_R} |w(x,t)| \leq \sup_{x\in B_R} (|u(x,t)| + |v(x,t)|) \leq ct^{-\frac{n}{\kappa}} \tag{3.1}$$

Let ζ be a standard cutoff function in B_{2R} with $\zeta \equiv 1$ in B_{2R} and $|\nabla\zeta| \le \frac{c}{R}$. Since $\lim_{t\to 0}[u(\cdot,t) - v(\cdot,t)] = 0$ in $L^1_{loc}(\mathbb{R}^n)$, we have

$$\begin{aligned}
&\int_{B_R} w(x,t)\zeta(x)dx \\
&\quad\le \int_0^t\int_{B_R} \left||\nabla u|^{p-1} + |\nabla v|^{p-1}\right| |\zeta|\, dxds + \int_0^t\int_{B_R} b(u^q+v^q)\zeta\, dxds \\
&\quad\le c\int_0^t\int_{B_R} \left(|\nabla u|^{p-1} + |\nabla v|^{p-1}\right)\, dxds + c\int_0^t\int_{B_R} b(u^q+v^q)\, dxds.
\end{aligned}$$

From Lemma 2.5 and Lemma 2.6 we also know that

$$\int_0^t\int_{B_R} |\nabla u|^{p-1} + |\nabla v|^{p-1}\, dxds \le c(N,p,n)t^{\frac{1}{\kappa}}$$

and

$$\int_0^t\int_{B_R} b(u^q+v^q)\, dxds \le c(N,R,p,n)t^{\alpha} \quad \text{for some} \quad 0<\alpha<1.$$

Since ζ is arbitrary, we get

$$\int_{B_R} |w(x,t)|\, dx \le c\left(t^{\frac{1}{\kappa}} + t^{\alpha}\right). \tag{3.2}$$

Combining (3.1) and (3.2) we have

$$\int_{B_R} |w(x,t)|^{1+\varepsilon}dx \le \sup|w(x,t)|^{\varepsilon}\int_{B_R}|w(x,t)|\, dx \le ct^{\frac{-\varepsilon n}{\kappa}}\left(t^{\frac{1}{\kappa}} + t^{\alpha}\right).$$

This completes the proof. □ □

Once we know the higher integrability lemma, we can prove uniqueness of nonnegative solutions. Here the Gronwall type inequality is established and hence uniqueness follows easily.

THEOREM 3.2. *Suppose u and v are two nonnegative weak solutions of (1.1) in $\mathbb{R}^n \times (0,T)$ for some $0<T<\infty$. If*

$$\sup_{t\in(0,T)}[\|u\|_1 + \|v\|_1] < \infty \quad \textit{and} \quad \lim_{t\to 0}[u(\cdot,t) - v(\cdot,t)] = 0$$

in $L^1_{loc}(\mathbb{R}^n)$, then $u \equiv v$ in $\mathbb{R}^n \times (0,t)$.

Proof. Let $w = u - v$ and we may assume $w \geq 0$. Let $\eta(x)$ be a standard cutoff function which is compactly supported in B_{R+1} and $\eta \equiv 1$ in B_R and $|\nabla \eta| < c$ for some constant c. We take $w^\varepsilon \eta^2$ as a test function to (1.1), then we obtain

$$\begin{aligned}
&\frac{1}{1+\varepsilon}\int_{B_R} w(x,t)^{1+\varepsilon}\eta^2\,dx - \frac{1}{1+\varepsilon}\int_{B_R} w(x,t)^{1+\varepsilon}\eta(x)^2\,dx \\
&+ \varepsilon\int_\tau^t\int_{B_R} \left(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\right)\nabla w\cdot w^{\varepsilon-1}\eta^2\,dxds \\
&+ 2\int_\tau^t\int_{B_R} \left(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\right) w^\varepsilon \eta\nabla\eta\,dxds \\
&+ \int_\tau^t\int_{B_R} b(u^q - v^q)\,dxds = 0.
\end{aligned}$$

From Lemma 3.1, we know that $\lim_{\tau\to 0}\int_{B_R} w(x,\tau)^{1+\varepsilon}\,dx = 0$. Letting $\tau \to 0$ and by Young's inequality we have

$$\begin{aligned}
&\frac{1}{1+\varepsilon}\int_{B_R} w(x,t)^{1+\varepsilon}\eta(x)^2\,dx \\
&\quad + \varepsilon\int_0^t\int_{B_R} (|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v)\nabla w w^{\varepsilon-1}\eta^2\,dxds \\
&\leq c\int_0^t\int_{B_{R+1}} (|\nabla u| + |\nabla v|)^{p-2}|\nabla u - \nabla v| w^\varepsilon\,dxds \\
&\leq \frac{c}{2}\int_0^t\int_{B_{R+1}} \left[\delta^{\frac{1}{2}}(|\nabla u| + |\nabla v|)^{\frac{p-2}{2}}|\nabla u - \nabla v| w^{\frac{\varepsilon-1}{2}}\right]^2 dxds \\
&\quad + \frac{c}{2}\int_0^t\int_{B_{R+1}} \left[\delta^{-\frac{1}{2}}(|\nabla u| + |\nabla v|)^{\frac{\varepsilon-1}{2}} w^{\frac{\varepsilon+1}{2}}\right]^2 dxds.
\end{aligned}$$

And

$$\begin{aligned}
&\int_{B_R} w(x,t)^{1+\varepsilon}\,dx \\
&\leq c\delta\int_0^t\int_{B_{R+1}} (|\nabla u| + |\nabla v|)^{p-2}|\nabla u - \nabla v|^2 w^{\varepsilon-1}\,dxds \\
&\quad + \frac{c}{\delta}\int_0^t\int_{B_{R+1}} (|\nabla u| + |\nabla v|)^{p-2} w^{\varepsilon+1}\,dxds.
\end{aligned} \tag{3.3}$$

We absorbs the integral involving $w^{\varepsilon-1}$ on the right hand side of (3.4). Then

$$\int_{B_R} w(x,t)^{1+\varepsilon}\,dx \le c\int_0^t\int_{B_{R+1}} (|\nabla u|+|\nabla v|)^{p-2} w^{\varepsilon+1}\,dxds \le$$

$$c\left[\int_0^t\int_{B_{R+1}} (|\nabla u|+|\nabla v|)^{p-1}\,dxds\right]^{\frac{p-2}{p-1}}\left[\int_0^t\int_{B_{R+1}} w^{(\varepsilon+1)(p-1)}\,dxds\right]^{\frac{1}{p-1}}.$$

From Lemma 2.5, we know that

$$\int_0^t\int (|\nabla u|+|\nabla v|)^{p-1}\,dxds \le ct^{-\frac{n}{\kappa}}.$$

On the other hand, from Lemma 2.4, $\sup w \le ct^{-\frac{n}{\kappa}}$. Hence we obtain that

$$\begin{aligned}\int_{B_R} w^{1+\varepsilon}\,dx &\le ct^{\frac{1}{\kappa}\frac{p-2}{p-1}}\left[\int_0^t\int_{B_{R+1}} w^{(\varepsilon+1)(p-2)} w^{(1+\varepsilon)}\,dxds\right]^{\frac{1}{p-1}}\\ &\le ct^{\frac{p-2}{\kappa(p-1)}}\left[\int_0^t \|w\|_\infty^{(p-2)(1+\varepsilon)}\int_{B_{R+1}} w^{1+\varepsilon}\,dxds\right]^{\frac{1}{p-1}}\\ &\le ct^{\frac{p-2}{\kappa(p-1)}}\left[\int_0^t s^{-\frac{n}{\kappa}(1+\varepsilon)(p-2)}\int_{B_{R+1}} w^{1+\varepsilon} dx\,ds\right]^{\frac{1}{p-1}}\end{aligned}$$

and

(3.4)
$$\begin{aligned}&t^{-\frac{p-2}{\kappa(p-1)}}\int_{B_R} w^{1+\varepsilon}\,dx\\ &\le c\left[\int_0^T s^{-\frac{n}{\kappa}(1+\varepsilon)(p-2)+\frac{p-2}{\kappa(p-1)}}\, s^{-\frac{p-2}{\kappa(p-1)}}\int_{B_{R+1}} w^{1+\varepsilon}\,dx\,ds\right]^{\frac{1}{p-1}}.\end{aligned}$$

Now we establish Grownwall type inequality. This implies uniqueness. Define

$$H(R,t) = t^{-\frac{p-2}{\kappa(p-1)}}\int_{B_R} w^{1+\varepsilon}\,dx.$$

Then (3.4) becomes

$$(3.5)\qquad H(R,t) \le c\left[\int_0^t s^{-\frac{n}{\kappa}(1+\varepsilon)(p-2)+\frac{p-2}{\kappa(p-1)}} H(R+1,s)\,ds\right]^{\frac{1}{(}p-1)}.$$

Let $\delta \equiv -\frac{n}{\kappa}(1+\varepsilon)(p-2)+\frac{p-2}{\kappa(p-1)}$, then $\delta > -1$ for small ε. Moreover we note that the smooth function $G : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ which is defined by

$$G(R) = \int_{B_R} w(x,t)^{1+\varepsilon} A_\alpha(x)\,dx$$

with $A_\alpha(x) \equiv (1+|x|^p)^{-\alpha}, \alpha p = \frac{\kappa}{p-2}+R$ satisfies the following properties

$$\frac{G(R+1)}{G(R)} \le c \text{ for some } c;\ \frac{H(R,t)}{G(R)} \to 0 \text{ as } R \to 0;$$
$$\frac{H(R,t)}{G(R)} \to 0 \text{ as } R \to \infty.$$

From this we can find R_1 such that $\frac{H(R_1,t)}{G(R_1)} = \sup_{R\ge 1} \frac{H(R,t)}{G(R)}$. And

$$(3.6)\qquad \frac{H(R,t)}{G(R)} \le c\frac{G(R+1)}{G(R)}\left[\int_0^t s^\delta \frac{H(R+1,s)}{G(R+1)\,ds}\right]^{\frac{1}{p-1}}$$

Let $\int_0^t s^\delta \sup_{R\ge 1} \frac{H(R,s)}{G(R)}\,ds = A(t)$. Then we get from (3.5)

$$\begin{aligned} A'(t)t^{-\delta} &\le c\left[\int_0^t s^\delta \frac{H(R_1+1,s)}{G(R_1+1)}\,ds\right]^{\frac{1}{p-1}} \\ &\le c\left[\int_0^t s^\delta \sup_{R\ge 1} \frac{H(R,s)}{G(R)}\,ds\right]^{\frac{1}{p-1}} \le cA(t). \end{aligned}$$

Since $\delta > -1$, t^δ is summable. By Gronwall inequality, $A(t) \equiv 0$. Therefore we conclude that $u \equiv v$. $\square$ $\square$

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