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CONFORMAL CHANGE OF THE TENSOR $U^{\nu}{}_{\lambda\mu}$ FOR THE SECOND CATEGORY IN 6-DIMENSIONAL g-UFT

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Abstract. We investigate chnage of the tensor $U^{\nu}{}_{\lambda\mu}$ induced by the conformal change in 6-dimensional g-unified field theory. These topics will be studied for the second class with the second category in 6-dimensional case.

1. Introduction

The conformal change in a generalied 4-dimensional Riemannian space connected by an Einstein's connection was primarily studied by HLAVATÝ([9], 1957), CHUNG ([7], 1968) also investigated the same topic in 4-dimensional $^*g$-unified field theory.

The Einstein's connection induced by the conformal change for all classes in 3-dimensional case and for the second and third classes in 5-dimensional case and for the first class in 5-dimensional case and for the second class with the first category in 6-dimensional case were investigated by CHO([1],1992, [2],1994, [3],1995, [4],1996).

In the present paper, we investigate change of the tensor $U^{\nu}{}_{\lambda\mu}$ induced by the conformal change in 6-dimensional g-unified field theory. These topics will be studied for the second class with the second category in 6-dimensional case.

2. Preliminaries

This chapter is a brief collection of basic concepts, notations, theorems, and results needed in our further considerations. They may

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be refferd to CHUNG([5],1982; [3],1988), CHO([1],1992; [2],1994; [3],1995; [4],1996).

2.1. n-dimensional g-unified field theory

The n-dimensional g-unified field theory (n-g-UFT hereafter) was originally suggested by HLAVATÝ([9],1957) and systematically introduced by CHUNG([8],1963).

Let X_n[1] be an n-dimensional generalized Riemannian manifold, referred to a real coordinate system x^ν obeying coordinate transformations $x^\nu \to x^{\nu'}$, for which

$$\text{Det}\left(\left(\frac{\partial x}{\partial x'}\right)\right) \neq 0. \tag{2.1}$$

In the usual Einstein's n-dimensional unified field theory, the manifold X_n is endowed with a general real nonsymmetric tensor $g_{\lambda\mu}$ which may be split into its symmetric part $h_{\lambda\mu}$ and skew-symmetric part $k_{\lambda\mu}$[2] :

$$g_{\lambda\mu} = h_{\lambda\mu} + k_{\lambda\mu} \tag{2.2}$$

where

$$\text{Det}((g_{\lambda\mu})) \neq 0, \qquad \text{Det}((h_{\lambda\mu})) \neq 0. \tag{2.3}$$

Therefore we may define a unique tensor $h^{\lambda\nu} = h^{\nu\lambda}$ by

$$h_{\lambda\mu} h^{\lambda\nu} = \delta^\nu_\mu. \tag{2.4}$$

In our n-g-UFT, the tensors $h_{\lambda\mu}$ and $h^{\lambda\nu}$ will serve for raising and/or lowering indices of the tensors in X_n in the usual manner.

The manifold X_n is connected by a general real connection $\Gamma^\nu_{\omega\mu}$ with the following transformation rule :

$$\Gamma^{\nu'}_{\omega'\mu'} = \frac{\partial x^{\nu'}}{\partial x^\alpha}\left(\frac{\partial x^\beta}{\partial x^{\omega'}} \cdot \frac{\partial x^\gamma}{\partial x^{\mu'}} \Gamma^\alpha_{\beta\gamma} + \frac{\partial^2 x^\alpha}{\partial x^{\omega'} \partial x^{\mu'}}\right) \tag{2.5}$$

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[1] Throughout the present paper, we assumed that $n \geq 2$.

[2] Throughout this paper, Greek indices are used for holonomic components of tensors. In X_n all indices take the values $1, \cdots, n$ and follow the summation convertion.

and satisfies the system of Einstein's equations

$$D_w g_{\lambda\mu} = 2S_{w\mu}{}^{\alpha} g_{\lambda\alpha} \tag{2.6}$$

where D_w denotes the covariant derivative with respect to $\Gamma^{\nu}_{\lambda\mu}$ and

$$S_{\lambda\mu}{}^{\nu} = \Gamma^{\nu}_{[\lambda\mu]} \tag{2.7}$$

is the *torsion tensor* of $\Gamma^{\nu}_{\lambda\mu}$. The connection $\Gamma^{\nu}_{\lambda\mu}$ satisfying (2.6) is called the *Einstein's connection.*

In our further considerations, the following scalars, tensors, abbreviations, and notations for $p = 0, 1, 2, \cdots$ are frequently used :

(2.8)a
$$\mathfrak{g} = \mathrm{Det}((g_{\lambda\mu})) \neq 0,\ \mathfrak{h} = \mathrm{Det}((h_{\lambda\mu})) \neq 0,\quad \mathfrak{k} = \mathrm{Det}((k_{\lambda\mu})),$$

(2.8)b
$$g = \frac{\mathfrak{g}}{\mathfrak{h}}, \quad k = \frac{\mathfrak{k}}{\mathfrak{h}},$$

(2.8)c
$$K_p = k_{[\alpha_1}{}^{\alpha_1} \cdots k_{\alpha_p]}{}^{\alpha_p}, \quad (p = 0, 1, 2, \cdots)$$

(2.8)d
$${}^{(0)}k_{\lambda}{}^{\nu} = \delta^{\nu}_{\lambda},\ {}^{(1)}k_{\lambda}{}^{\nu} = k_{\lambda}{}^{\nu},\ {}^{(p)}k_{\lambda}{}^{\nu} = {}^{(p-1)}k_{\lambda}{}^{\alpha} k_{\alpha}{}^{\nu},$$

(2.8)e
$$K_{\omega\mu\nu} = \nabla_{\nu} k_{\omega\mu} + \nabla_{\omega} k_{\nu\mu} + \nabla_{\mu} k_{\omega\nu},$$

(2.8)f
$$\sigma = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}.$$

where ∇_{ω} is the symbolic vector of the convariant derivative with respect to the Christoffel symbols $\{^{\ \nu}_{\lambda\mu}\}$ defined by $h_{\lambda\mu}$. The scalars and vectors introduced in (2.8) satisfy

(2.9)a
$$K_0 = 1;\ K_n = k \text{ if } n \text{ is even};\ K_p = 0 \text{ if } p \text{ is odd},$$

(2.9)b
$$g = 1 + K_2 + \cdots + K_{n-\sigma},$$

(2.9)c
$$^{(p)}k_{\lambda\mu} = (-1)^{p(p)}k_{\mu\lambda}, \quad ^{(p)}k^{\lambda\nu} = (-1)^{p(p)}k^{\nu\lambda}.$$

Furthermore, we also use the following useful abbreviations, denoting an arbitrary tensor $T_{\omega\mu\nu}$, skew-symmetric in the first two indices, by T :

(2.10)a
$$\overset{pqr}{T} = \overset{pqr}{T}{}_{\omega\mu\nu} = T_{\alpha\beta\gamma}{}^{(p)}k_\omega{}^{\alpha(q)}k_\mu{}^{\beta(r)}k_\nu{}^\gamma,$$

(2.10)b
$$T = T_{\omega\mu\nu} = \overset{000}{T},$$

(2.10)c
$$2\overset{pqr}{T}{}_{\omega[\lambda\mu]} = \overset{pqr}{T}{}_{\omega\lambda\mu} - \overset{pqr}{T}{}_{\omega\mu\lambda},$$

(2.10)d
$$2\overset{(pq)r}{T}{}_{\omega\lambda\mu} = \overset{pqr}{T}{}_{\omega\lambda\mu} + \overset{qpr}{T}{}_{\omega\lambda\mu}.$$

We then have

(2.11)
$$\overset{pqr}{T}{}_{\omega\lambda\mu} = -\overset{qpr}{T}{}_{\lambda\omega\mu}.$$

If the system (2.6) admits $\Gamma^\nu_{\lambda\mu}$, using the above abbreviations it was shown that the connection is of the form

(2.12)
$$\Gamma^\nu_{\omega\mu} = \{{}^{\nu}_{\omega\mu}\} + S_{\omega\mu}{}^\nu + U^\nu{}_{\omega\mu}$$

where

(2.13)
$$U_{\nu\omega\mu} = \overset{100}{S}{}_{(\omega\mu)\nu}.$$

The above two relations show that *our problem of determining* $\Gamma^\nu_{\omega\mu}$ *in terms of* $g_{\lambda\mu}$ *is reduced to that of studying the tensor* $S_{\omega\mu}{}^\nu$. On the other hand, it has also been shown that the tensor $S_{\omega\mu}{}^\nu$ satisfies

(2.14)
$$S = B - 3\overset{(110)}{S}$$

where

(2.15)
$$2B_{\omega\mu\nu} = K_{\omega\mu\nu} + 3K_{\alpha[\mu\beta}k_{\omega]}{}^\alpha k_\nu{}^\beta.$$

2.2. Some results in 6-g-UFT

In this section, we introduce some results 6-g-UFT without proof, which are needed in our subsequent considerations. They may be referred to CHO([5],1993).

DEFINITION 2.1. In 6-g-UFT, the tensor $g_{\lambda\mu}(k_{\lambda\mu})$ is said to be the second class with the second categoty, if $K_4 \neq 0$, $K_6 = 0$.

THEOREM 2.2. *(Main recurrence relations) For the second class with the second category in 6-UFT, the following recurrence relation hold*

$$ {}^{(p+4)}k_\lambda{}^\nu = -K_2\,{}^{(p+2)}k_\lambda{}^\nu - K_4\,{}^{(p)}k_\lambda{}^\nu, \quad (p = 0, 1, 2, \cdots) \tag{2.16}$$

THEOREM 2.3. *(For the second class with the second category in 6-g-UFT). A necessary and sufficient condition for the existence of the solution of (2.5) is*

$$ (1 + K_2 + K_4)[(1 - K_2 + 5K_4)^2 - 4K_4(2 - K_2)^2] \neq 0. \tag{2.17}$$

3. Conformal change of the 6-dimensional tensor $U^\nu{}_{\lambda\mu}$ for the second class with the second category.

In this final chapter we investigate the change $U^\nu{}_{\lambda\mu} \to \overline{U}^\nu{}_{\lambda\mu}$ of the tensor induced by the conformal change of the tensor $g_{\lambda\mu}$, using the recurrence relations and theorems introduced in the preceding chapter.

We say that X_n and $\overline{X}_n$ are conformal if and only if

$$ \overline{g}_{\lambda\mu}(x) = e^\Omega g_{\lambda\mu}(x) \tag{3.1}$$

where $\Omega = \Omega(x)$ is an at least twice differentiable function. This conformal change enforces a change of the tensor $U^\nu{}_{\lambda\mu}$. An explicit representation of the change of 6-dimensional tensor $U^\nu{}_{\lambda\mu}$ for the second class with the second category will be exhibited in this chapter.

Agreement 3.1. Throughout this section, we agree that, if T is a function of $g_{\lambda\mu}$, then we denote $\overline{T}$ the same function of $\overline{g}_{\lambda\mu}$. In particular, if T is a tensor, so is $\overline{T}$. Furthermore, the indices of T ($\overline{T}$) will be raised and/or lowered by means of $h^{\lambda\nu}$ ($\overline{h}^{\lambda\nu}$) and/or $h_{\lambda\mu}$ ($\overline{h}_{\lambda\mu}$).

The results in the following theorems needed in our further considerations. They may be referred to CHO([1],1992, [2],1994, [3],1995, [4],1996).

THEOREM 3.2. *In n-g-UFT, the conformal change (3.1) induces the following changes :*

$$\begin{aligned} {}^{(p)}\overline{k}_{\lambda\mu} &= e^{\Omega(p)} k_{\lambda\mu}, \quad {}^{(p)}\overline{k}_{\lambda}{}^{\nu} = {}^{(p)}k_{\lambda}{}^{\nu}, \\ {}^{(p)}\overline{k}^{\lambda\nu} &= e^{-\Omega(p)} k^{\lambda\nu} \end{aligned} \tag{3.2a}$$

$$\overline{g} = g, \quad \overline{K_p} = K_p, \qquad (p = 1, 2, \cdots). \tag{3.2b}$$

Now, we are ready to derive representations of the changes $U^{\nu}{}_{\omega\mu} \rightarrow \overline{U}^{\nu}{}_{\omega\mu}$ in 6-g-UFT for the second class with the second category induced by the conformal change (3.1).

THEOREM 3.3. *The change $S_{\omega\mu}{}^{\nu} \rightarrow \overline{S}_{\omega\mu}{}^{\nu}$ induced by conformal change (3.1) may be represented by*

$$\begin{aligned} \overline{S}_{\omega\mu}{}^{\nu} = & S_{\omega\mu}{}^{\nu} + \frac{1}{C}[a_1 k_{\omega\mu}\Omega^{\nu} + a_2 k^{\nu}{}_{[\omega}\Omega_{\mu]} \\ & + a_3 h^{\nu}{}_{[\omega}k_{\mu]}{}^{\delta}\Omega_{\delta} + a_4 \delta^{\nu}{}_{[\omega}k_{\mu]} \\ & + a_5 k^{\nu}{}_{[\omega}{}^{(2)}k_{\mu]}{}^{\delta}\Omega_{\delta} + a_6{}^{(2)}k^{\nu}{}_{[\omega}k_{\mu]}{}^{\delta}\Omega_{\delta} \\ & + a_7 k_{\omega\mu}{}^{(2)}k^{\nu\delta}\Omega_{\delta} + a_8{}^{(3)}k^{\nu}{}_{[\omega}\Omega_{\mu]} \\ & + a_9{}^{(3)}k^{\nu}{}_{[\omega}\Omega_{\mu]} + a_{10}\delta^{\nu}{}_{[\omega}{}^{(3)}k_{\mu]}{}^{\delta}\Omega_{\delta} \\ & + 2a_{11}{}^{(3)}k^{\nu}{}_{[\omega}{}^{(2)}k_{\mu]}{}^{\delta}\Omega_{\delta} + 2a_{12}{}^{(2)}k^{\nu}{}_{[\omega}{}^{(3)}k_{\mu]}{}^{\delta}\Omega_{\delta} \\ & + a_{13}{}^{(3)}k_{\omega\mu}{}^{(2)}k^{\nu\delta}\Omega_{\delta]}, \end{aligned} \tag{3.3}$$

where

$$
\begin{aligned}
a_1 &= \alpha^2\beta(1+4\beta) - 2\alpha\beta(1+\beta+2\beta^2) + \beta(1-13\beta^2) - C,\\
a_2 &= 2\alpha^3\beta - \alpha^2\beta(1-2\beta) + 2\alpha\beta^2(1-2\beta) + \beta^2(3\beta-4) + C,\\
a_3 &= \beta^2(2\alpha^2 - 5\alpha - 9\beta + 7) - C,\\
a_4 &= -2\alpha^3\beta + \alpha^2\beta(1+12\beta) - 9\alpha\beta^2 - \beta(3+5\beta+18\beta^2),\\
a_5 &= 2\alpha^4 - \alpha^3(2\beta+3) - \alpha^2(1+9\beta+4\beta^2)\\
&\quad + \alpha(2-10\beta-\beta^2+8\beta^3) + \beta(6+13\beta+19\beta^2),\\
a_6 &= -2\alpha^4 + \alpha^3(1+18\beta) + 2\alpha^2\beta(1-8\beta) - \alpha(2+16\beta\\
&\quad + 59\beta^2 + 8\beta^3) + \beta(27\beta^2 - 58\beta - 10) - 1 + 2C,
\end{aligned}
$$

$$
\begin{aligned}
a_7 &= -\alpha^2\beta(1+4\beta) + 2\alpha\beta(1+\beta) + \beta(13\beta^2 + 4\alpha\beta^2 - 1) + C,\\
a_8 &= 3\alpha^3 + \alpha^2(5\beta + 8\beta^2 - 4) - \alpha(2+36\beta+5\beta^2)\\
&\quad + 7\beta(2-6\beta) - 3\beta^2) + 3,\\
a_9 &= \alpha^2(1-8\beta) - 2\alpha(1-6\beta^2) + \beta(8\beta^2 + 35\beta - 12) + 1,\\
a_{10} &= 2\alpha^2\beta(-5+2\beta) + 2\alpha\beta(3-6\beta+4\beta^2) + 4\beta(1+2\beta-2\beta^2),\\
a_{11} &= 2\alpha^4 - \alpha^3(1+3\beta) - 4\alpha^2\beta^2 + \alpha(1+7\beta+4\beta^2)\\
&\quad - \beta(3-7\alpha-4\alpha\beta) - 2,\\
a_{12} &= 2\alpha^4 + \alpha^3(2\beta-15) + \alpha^2(22-19\beta+4\beta^2)\\
&\quad + \alpha(-8+35\beta-6\beta^2) - 3\beta + 1,\\
a_{13} &= -4\alpha^4 - \alpha^3(1-8\beta) + 11\alpha^2\beta - \alpha(8-16\beta+21\beta^2)\\
&\quad + \beta(5\beta^2 + 2\beta - 10) - 3,
\end{aligned}
$$

where $\alpha = K_2$, $\beta = K_4$,

$$C = (1+\alpha+\beta)[(1-\alpha+5\beta)^2 - 4\beta(2-\alpha)^2]. \tag{3.4}$$

THEOREM 3.4. *The change* $U^{\nu}{}_{\omega\mu} \to \overline{U}^{\nu}{}_{\omega\mu}$ *induced by the con-*

formal change (3.1) may be represented by

$$\overline{U}^\nu{}_{\omega\mu} = U^\nu{}_{\omega\mu} + \frac{1}{C}[b_1 \delta^\nu{}_{(\omega}\Omega_{\mu)} + b_2 {}^{(2)}k^\nu{}_{(\omega}\Omega_{\mu)}$$
$$+ (b_3 \delta^\nu{}_{(\omega}{}^{(2)}k_{\mu)}{}^\delta + b_4 k^\nu{}_{(\omega}k_{\mu)}{}^\delta$$
$$(3.5) \qquad + b_5 {}^{(2)}k^\nu{}_{(\omega}k_{\mu)}{}^\delta + b_6 k^\nu{}_{(\omega}{}^{(3)}k_{\mu)}{}^\delta$$
$$+ b_7 {}^{(2)}k^\nu{}_{(\omega}{}^{(2)}k_{\mu)}{}^\delta + b_8 {}^{(3)}k^\nu{}_{(\omega}k_{\mu)}{}^\delta$$
$$+ b_9 {}^{(2)}k^\nu{}_{(\omega}{}^{(3)}k_{\mu)}{}^\delta + b_{10} {}^{(3)}k^\nu{}_{(\omega}{}^{(3)}k_{\mu)}{}^\delta)\Omega_\delta],$$

where

$$b_1 = \beta[\alpha^2(8\beta - 1) - 2\alpha(6\beta^2 - 1) - \beta(8\beta^2 + 35\beta - 12) - 1],$$
$$b_2 = \alpha[\alpha^2(11\beta - 1) - \alpha(10\beta^2 + \beta - 2) - 12\beta^3 - 33\beta^2 + 12\beta - 1] + \beta^3(3\beta - 4) + C,$$
$$b_3 = \alpha\beta[-2\alpha^3 + \alpha^2(1 + 3\beta) + 4\alpha\beta^2 - 1 - 14\beta - 8\beta^2] + 2\beta,$$
$$b_4 = \beta[-2\alpha^2(\alpha + \beta + 3) - 2\alpha\beta(2\beta + 3) - 29\beta^2 - 4\beta - 2] - 3C,$$
$$b_5 = \beta[-2\alpha^3 + \alpha^2(12\beta + 1) - 9\alpha\beta - 18\beta^2 - 5\beta - 3],$$
$$b_6 = \beta[-4\alpha^2(6\beta - 4) + 2\alpha(14\beta - 1) + 34\beta^2 - 8\beta - 6] + 2C,$$
$$b_7 = \alpha[-2\alpha^4 + 3\alpha^3(\beta + 1) + 4\alpha^2\beta^2 - \alpha(12\beta^2 + 23\beta + 2) + 8\beta^3 - \beta^2 - 7\beta + 4] + \beta(19\beta^2 + 13\beta + 6),$$
$$b_8 = \alpha[2\alpha^3 - 18\alpha^2\beta + 5\alpha^2 + 32\alpha\beta^2 + 8\alpha\beta - 8\alpha + 8\beta^3 + 49\beta^2 - 56\beta - 2] - \beta(27\beta^3 + 42\beta^2 + 26\beta - 38) + 1 - 2C,$$
$$b_9 = 2\beta[\alpha^2(2\beta - 5) + \alpha(4\beta^2 - 6\beta + 3) + 2(-2\beta^2 + 2\beta + 1)],$$
$$b_{10} = \alpha[-10\alpha^3 + \alpha^2(14\beta + 13) + 41\alpha\beta - 2\alpha(2\beta + 11) - 35\beta^2 - 3\beta - 8] + \beta(10\beta^2 + 4\beta - 17) - 7,$$

where $\alpha = K_2$, $\beta = K_4$.

Proof. In virtue of (2.13) and Agreement (3.1), we have

$$(3.6) \qquad \overline{U}_{\nu\omega\mu} = \overline{\overset{100}{S}}_{(\omega\mu)\nu},$$

The relation (3.5) follows by substituting (3.3), (2.10), Definition (2.1), (2.16), (3.2) into (3.6). □ □

References

[1] C.H. Cho, *Conformal change of the connection in 3- and 5-dimensional* $^*g^{\lambda\nu}$*-Unified Field Theory*, BIBS, Inha Univ. **13** (1992).

[2] C.H. Cho, *Conformal change of the connection for the first class in 5-dimensional* $^*g^{\lambda\nu}$*-unified Field Theory*, Kangweon-Kyungki Math. Jour. **2** (1994), 19–33.

[3] C.H. Cho, *Conformal change of the torsion tensor in 6-dimensional g-unified field theory,*, Kangweon-Kyungki Math. Jour. **3, No.1** (1995), 71–79.

[4] C.H. cho, *conformal change of the tensor* $S_{\lambda\mu}{}^{\nu}$ *for the second category in 6-dimensional g-UFT*, Kangweon-Kyungki Math. Jour. **4, No.2** (1996), 163–171.

[5] C.H. Cho and G.T. Yang, *On the recurrence relations in* X_6, BIBS, Inha Univ. **14** (1993), 13–22.

[6] K.T. Chung, *Three- and Five-dimensional considerations of the geometry of Einstein's* $^*g$*-unified field theory*, International Journal of Theoretical Physics **27(9)** (1988).

[7] K.T. Chung, *Conformal change in Einstein's* $^*g^{\lambda\nu}$*-unified field theory*, Nuove Cimento **(X) 58B** (1968).

[8] K.T. Chung, *Einstein's connection in terms of* $^*g^{\lambda\nu}$, Nuovo Cimento **(X) 27** (1963).

[9] V. HLAVATÝ, *Geometry of Einstein's unified field theory*, P. Noordhoff Ltd. (1957).

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