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HARMONIC LITTLE BLOCH FUNCTIONS ON THE UPPER HALF-SPACE

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Abstract. On the setting of the upper half-space of the euclidean n-space, we study some properties of harmonic little Bloch functions and we show that for a given harmonic little Bloch function u, there exists unique harmonic conjugates of u which are also little Bloch functions with appropriate norm bounds.

1. Introduction

The upper half-space $H = H_n$ is the open subset of $\mathbf{R}^n (n \geq 2)$ given by

$$H = \{z = (z', z_n) \in \mathbf{R}^n : \; z_n > 0\},$$

where we have written a typical point $z \in \mathbf{R}^n$ as $z = (z', z_n)$, with $z' \in \mathbf{R}^{n-1}$ and $z_n \in \mathbf{R}$.

Given a harmonic function u on H, the functions $v_1, \dots, v_{n-1}$ on H are called harmonic conjugates of u if

$$(v_1, \dots, v_{n-1}, u) = \nabla f \tag{1.1}$$

for some harmonic function f on H, where ∇f denotes the gradient of f. If (1.1) holds, then $v_1, \dots, v_{n-1}$ are partial derivatives of a harmonic function, so they are harmonic functions on H. Also (1.1) and the

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condition that f is harmonic is equivalent to the following "generalized Cauchy-Riemann equations"

$$D_k v_j = D_j v_k; D_n v_j = D_j u$$
$$\sum_{j=1}^{n-1} D_j v_j + D_n u = 0.$$

In particular, v is a harmonic conjugate of u if and only if $u + iv$ is a holomorphic function on the upper half-plane H_2.

If u is harmonic function on H, then harmonic conjugates of u always exist. However they are far from unique. (If $n > 2$, then harmonic conjugates of a given harmonic function u may well differ more than a constant. We refer more on these to [3].)

A harmonic function u on H is called a harmonic Bloch function if

$$\|u\|_{\mathcal{B}} = \sup z_n |\nabla u(z)| < \infty,$$

where the supremum is taken over all $z \in H$. (Here we use the $\mathbf{C}^n$ norm to calculate $|\nabla u(z)|$.) We let $\widetilde{\mathcal{B}}$ denote the collection of harmonic Bloch functions that vanish at $z_0 = (0, 1)$. Then we can show that $\widetilde{\mathcal{B}}$ is a Banach space under the Bloch norm $\| \ \|_{\mathcal{B}}$.

A harmonic Bloch function u is called a harmonic little Bloch function if it has the following vanishing condition;

$$\lim_{z \to \partial^\infty H} z_n |\nabla u(z)| = 0, \tag{1.2}$$

where $\partial^\infty H$ denotes the union of ∂H and $\{\infty\}$. We let $\widetilde{\mathcal{B}}_0$ denote the set of all harmonic little Bloch functions on H vanishing at z_0. Then we can show that $\widetilde{\mathcal{B}}_0$ is also a Banach space under $\| \ \|_{\mathcal{B}}$ from a straightfoward computation. If we let $C_0(H)$ denote the set of all continuous functions on H vanishing at ∞, then it is easy to show that the condition (1.2) is equivalent to the condition that the function $u \in \widetilde{\mathcal{B}}$ satisfies

$$z_n |\nabla u(z)| \in C_0(H). \tag{1.3}$$

In this paper we show some properties of harmonic little Bloch functions and we prove that for a given $u \in \widetilde{\mathcal{B}}_0$ there exist unique harmonic conjugates of u in $\widetilde{\mathcal{B}}_0$ with appropriate norm bounds. (In [4], there is a corresponding result to harmonic Bloch functions.)

2. Preliminary

In this section, we review some preliminary results from [1], [2]. First let's recall that b^2 is the harmonic Bergman space consisting of all harmonic functions u on H such that

$$\|u\|_2 = \left(\int_H |u|^2 \, dV \right)^{1/2} < \infty$$

where dV denotes the Lebegue volume measure on H, which we may write dz, dw, etc. By the mean value property and Jensen's inequality, one can easily verify that

$$|u(z)|^2 \le \sigma_n^{-1} z_n^{-n} \|u\|_2^2 \tag{2.1}$$

holds for all $u \in b^2$ and for every $z \in H$. Here we use the notation σ_n for the volume of the unit ball of $\mathbf{R}^n$. It follows from inequality (2.1) that norm convergence in b^2 implies uniform convergence on compact subsets of H. Thus, b^2 is a Hilbert space. Inequality (2.1) also gives that, for each fixed $z \in H$, the map $z \mapsto u(z)$ is a bounded linear functional on b^2 and hence there exists a unique function $R(z, \cdot) \in b^2$, called the harmonic Bergman kernel, such that

$$u(z) = \int_H u(w) \overline{R(z,w)} \, dw$$

for all $u \in b^2$. It is known that $R(z,w) = R(w,z)$ and that $R(z,w)$ is real valued; thus we can remove the complex conjugate in the integral above. For this and related results see Chapter 8 of [1]. The explicit formula for the harmonic Bergman kernel is given by

$$R(z,w) = \frac{4}{n\sigma_n} \frac{n(z_n + w_n)^2 - |z - \overline{w}|^2}{|z - \overline{w}|^{n+2}}.$$

Here we use the notation $\overline{w} = (w', -w_n)$ for $w \in H$. Note that if $n = 2$, then $\overline{w}$ is the usual complex conjugate of w. From this explicit formula for $R(z,w)$, we can show that for each fixed $z \in H$, this harmonic Bergman kernel $R(z, \cdot)$ is not even integrable. Since every bounded harmonic function on H is a harmonic Bloch function, one can not expect that $R(z,w)$ has a reproducing property for harmonic Bloch functions, however by modifying $R(z,w)$ to $\widetilde{R}(z,w)$ by

$$\widetilde{R}(z,w) = R(z,w) - R(z_0,w),$$

one can see that $\widetilde{R}(z,\cdot)$ is integrable on H and it satisfies the following reproducing property: If $u \in \widetilde{\mathcal{B}}$, then for every $z \in H$,

$$\begin{aligned} u(z) &= \int_H u(w)\widetilde{R}(z,w)\,dw \\ &= -2\int_H [D_{w_n}u(w)]w_n\widetilde{R}(z,w)\,dw \\ &= -2\int_H u(w)w_n D_{w_n}\widetilde{R}(z,w)\,dw. \end{aligned} \tag{2.2}$$

From the explicit formula of $\widetilde{R}(z,w)$, we can easily check that there is a positive constant C depending only on n such that

$$|\nabla_z\widetilde{R}(z,w)| = |\nabla_z R(z,w)| \le \frac{C(n)}{|z-\overline{w}|^{n+1}} \tag{2.3}$$

for all $z, w \in H$.

In [2], it is known that the Bloch norm is equivalent to the normal derivative norm: there are two positive constants c and C depending only on n such that

$$c\|u\|_{\mathcal{B}} \le \|z_n D_{z_n}u\|_\infty \le C\|u\|_{\mathcal{B}} \tag{2.4}$$

for all $u \in \widetilde{\mathcal{B}}$.

3. Main results

In this section we prove the main results of this paper. For $u \in \widetilde{\mathcal{B}}_0$ and for $j = 1, 2, \ldots, n-1$, set

$$A_j[u](z) = -2\int_H [D_{w_j}u(w)]w_n\widetilde{R}(z,w)\,dw \tag{3.1}$$

for every $z \in H$. Then by taking the integration by parts with respect to w_j-axis, we easily get

$$A_j[u](z) = 2\int_H u(w)w_n D_{w_j}\widetilde{R}(z,w)\,dw \tag{3.2}$$

for $z \in H$. Below we show that for each $j = 1, 2, \ldots, n-1$, A_j maps $\widetilde{\mathcal{B}}_0$ into $\widetilde{\mathcal{B}}_0$ boundedly. To do so we need the following lemma.

LEMMA 3.1. *There is a constant C depending only on n such that*

$$\int_H \frac{1}{|z-\overline{w}|^{n+1}}\,dw \leq \frac{C}{z_n}$$

for all $z \in H$.

Proof. Fix $z \in H$. Then we have

$$\int_H \frac{1}{|z-\overline{w}|^{n+1}}\,dw \leq \int_0^\infty \frac{1}{(z_n+w_n)^2}\int_{\mathbf{R}^{n-1}} \frac{(z_n+w_n)}{|z-\overline{w}|^n}\,dw'\,dw_n. \tag{3.3}$$

From the Poisson integral theory, we know the inner integral of (3.3) equals $n\sigma_n/2$. (See [1] and [3] for details on this.) Hence, after applying change of variable $w_n \longmapsto z_n w_n$, we see that

$$\int_H \frac{1}{|z-\overline{w}|^{n+1}}\,dw \leq \frac{C}{z_n}.$$

This completes the proof. □ □

We are now ready to show one of the main results of this paper. Here and for the rest of this paper, C denotes the constant depending only on n which varies from line to line.

THEOREM 3.2. *For each $j = 1, 2, \ldots, n-1$, the map A_j is bounded and linear from $\widetilde{\mathcal{B}}_0$ into $\widetilde{\mathcal{B}}_0$.*

Proof. Fix j. The linearlity of the map A_j is clear and let $u \in \widetilde{\mathcal{B}}_0$. Because $\widetilde{R}(z_0, w) = 0$, we have $A_j[u](z_0) = 0$. By passing the Laplacian $\triangle_z$ through the integral in (3.1), we easily see that $A_j[u]$ is harmonic on H since $\widetilde{R}(z,w)$ is harmonic as a function of z. Note that

$$\begin{aligned} z_n|\nabla A_j[u](z)| &= 2z_n\left|\int_H [D_{w_j}u(w)]w_n\nabla_z\widetilde{R}(z,w)\,dw\right| \\ &\leq Cz_n\|u\|_{\mathcal{B}}\int_H \frac{1}{|z-\overline{w}|^{n+1}}\,dw \\ &\leq C\|u\|_{\mathcal{B}}, \end{aligned}$$

where we used the estimate (2.3) and the Lemma 3.1. This shows that $A_j[u] \in \widetilde{\mathcal{B}}$ with $\|A_j[u]\|_{\mathcal{B}} \le C\|u\|_{\mathcal{B}}$. Therefore it remains to show the vanishing property (1.3) of $A_j[u]$. Note that

$$z_n|\nabla A_j[u](z)| \le C z_n \int_H |D_{w_j} u(w)| w_n \frac{1}{|z-\overline{w}|^{n+1}}\, dw$$

for some constant C. Note also that $w_n|D_{w_j}u(w)| \in C_0(H)$, because $u \in \widetilde{\mathcal{B}}_0$. Let $\epsilon > 0$. Then there is a compact set K in H such that $|w_n D_{w_j} u(w)| < \epsilon$ on $H \setminus K$. Therefore we have

$$\begin{aligned} z_n|\nabla A_j[u](z)| \le \epsilon C &\int_{H\setminus K} \frac{z_n}{|z-\overline{w}|^{n+1}}\, dw \\ &+ C\|u\|_{\mathcal{B}} \int_K \frac{z_n}{|z-\overline{w}|^{n+1}}\, dw. \end{aligned} \tag{3.4}$$

Let *I* and *II* denote, respectively, the two integrals of (3.4). Then from Lemma 3.1, we get

$$I \le \int_H \frac{z_n}{|z-\overline{w}|^{n+1}}\, dw \le C.$$

Notice that

$$II \le C(n,K)\frac{z_n}{1+|z|^{n+1}}$$

for some constant $C(n,K)$ depending only on n and K and notice also that the function $z \longmapsto z_n/(1+|z|^{n+1})$ is in $C_0(H)$. This shows that $z_n|\nabla A_j[u](z)| \in C_0(H)$ and the proof is complete. □ □

Now we are ready to prove that for a given $u \in \widetilde{\mathcal{B}}_0$, there are unique harmonic conjugates $v_1, v_2, \ldots, v_{n-1}$ of u which are also in $\widetilde{\mathcal{B}}_0$. The proof of the following theorem is quite similar to the proof of the theorem given in [4] for harmonic Bloch functions, however we give the proof of it for the reader's convenience.

THEOREM 3.3. *For each $u \in \widetilde{\mathcal{B}}_0$, there exist unique harmonic conjugates $v_1, \ldots, v_{n-1}$ of u on H such that $v_j \in \widetilde{\mathcal{B}}_0$ for each j. Moreover, there exists a positive constant C such that $\|v_j\|_{\mathcal{B}} \le C\|u\|_{\mathcal{B}}$ for each j.*

Proof. Let $u \in \widetilde{\mathcal{B}}_0$. For each $j = 1, \ldots, n-1$, set $v_j = A_j[u]$. Then by Theorem 3.2, we know each $v_j \in \widetilde{\mathcal{B}}_0$ and $\|v_j\|_{\mathcal{B}} \le C\|u\|_{\mathcal{B}}$ for each j. Now from the explicit formula of $\widetilde{R}$, we can check easily that for $j, k = 1, 2, \ldots, n-1$,

$$D_{z_k} D_{w_j} \widetilde{R}(z,w) = -D_{z_j} D_{z_k} R(z,w) = D_{z_j} D_{w_k} \widetilde{R}(z,w),$$

$$D_{z_n} D_{w_j} \widetilde{R}(z,w) = -D_{z_j} D_{w_n} \widetilde{R}(z,w). \tag{3.5}$$

Note that

$$D_{z_n} D_{w_n} \widetilde{R}(z,w) = D^2_{z_n} R(z,w).$$

Therefore by differentiating through the integral in (3.2), we have $D_k v_j = D_j v_k$. Similary from (3.1) and (3.5), we get $D_n v_j = D_j u$. Furthermore, from (2.2) and (3.2) we also have

$$\sum_{j=1}^{n-1} D_j v_j(z) + D_n u(z) = -2 \int_H u(w) w_n \triangle_z R(z,w)\, dw = 0$$

for all $z \in H$ since $R(z,w)$ is harmonic on H as a function of z. Therefore $v_1, \ldots, v_{n-1}, u$ satisfy the generalized Cauchy-Riemann equations and it follows that $v_1, \ldots, v_{n-1}$ are harmonic conjugates of u.

To complete the proof of this theorm, we only need to show the uniqueness part. Suppose that $u_1, \ldots, u_{n-1}$ are also harmonic conjugates of u satisfying $u_j \in \widetilde{\mathcal{B}}_0$ for each j. Then by (2.4) we have

$$\|v_j - u_j\|_{\mathcal{B}} \le C\|z_n D_{z_n}(v_j - u_j)\|_\infty$$

for each j. Because $D_{z_n}(v_j - u_j) = D_j(u - u) = 0$, we have $\|v_j - u_j\|_{\mathcal{B}} = 0$ and so $v_j = u_j$ for each j. This completes the proof. □ □

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