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BASIC CONSTRUCTIONS FOR $N_f \subset M_f$

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Abstract. We show that there exists an isomorphism between the basic construction $(M_f)_1$ for $N_f \subset M_f$ and the reduction $(M_1)_f$ of the basic construction M_1 for $N \subset M$, where f is a nontrivial projection in N. For a nontrivial projection $f \in N' \cap M$ we give the basic construction $(M_f)_1$ for $N_f \subset M_f$ and compare it with $(M_1)_f$.

1. Introduction

Murray and von Neumann defined the coupling constant of II_1-factor which measures the relative mobility of the factor and its commutant. Index theory for II_1-subfactors was introduced by Jones in [4] by using the coupling constants and it was extended by H. Kosaki in [5] to an arbitrary factors.

Jones' index theory is one of the most important and interesting topics in recent operator algebras and many connections with other areas of mathematics and mathematical physics are pointed out. Ocneau's paragroup theory, bimodule theory, and sector theory were introduced for the research of index theory [2,3,6,7,8].

We fix some notations and recall the definition of Jones' index. Let M be a finite von Neumann algebra with faithful normal normalized trace τ and N a von Neumann subalgebra of M. Then there exists a conditional expectation $E_N : M \to N$ defined by the relation $\tau(E_N(x)y) = \tau(xy)$, for $x \in M,\ y \in N$. If M is a finite factor acting on a Hilbert space H with finite commutant M', then the coupling constant $\dim_M(H)$ of M is defined as $\tau([M'\xi])/\tau'([M\xi])$, where $\xi_{\neq 0} \in H$, and τ' is a trace in M'. $L^2(M,\tau)$ is the Hilbert space of GNS representation of M given by τ and M acts on $L^2(M,\tau)$ by left

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multiplication. E_N extends to a projection e_N via $e_N(x\xi) = E_N(x)\xi$, where ξ is the canonical cyclic trace vector in $L^2(M, \tau)$. For a pair of finite factors $N \subset M$, Jones defined in [4] the index of N in M by $[M : N] = \dim_N(H)/\dim_M(H)$, or equivalently, $\dim_N(L^2(M, \tau))$. From now on, $N \subset M$ denotes a pair of II_1-factors with faithful normal normalized trace τ and finite Jones' index and M_1 the basic construction for $N \subset M$, which is generated by M and e_N.

In this paper, we study Jones' index for a pair of reduced II_1-subfactors $N_f \subset M_f$, where f is a nontrivial projection in N. We will prove that the basic construction $(M_f)_1$ for $N_f \subset M_f$ is isomorphic to $(M_1)_f$. We also study the Jones' index for induced and reduced II_1-subfactors $N_f \subset M_f$, where f is a nontrivial projection in $N' \cap M$. We will construct the basic construction $(M_f)_1$ for $N_f \subset M_f$ and compare $(M_f)_1$ with $(M_1)_f$.

2. The basic construction for reduced factors

For a nontrivial projection $f \in N$, if we consider the reductions $M_f = \{fxf|_{fH} | \ x \in M\}$ and $N_f = \{fxf|_{fH} | \ x \in N\}$, where M acts on H, then $N_f \subset M_f$ is a pair of II_1-factors. Since for a projection e in M $\dim_{M_e}(eH) = \tau(e)^{-1}\dim_M(H)$ holds, we have the following proposition.

Proposition 2.1. *If f is a nontrivial projection in N, then we have* $[M_f : N_f] = [M : N]$.

Proof. For a nontrivial projection f in N

$$\begin{aligned}[M_f : N_f] &= dim_{N_f}(fH)/dim_{M_f}(fH) \\ &= \tau(f)^{-1}dim_N(H)/\tau(f)^{-1}dim_M(H) = [M : N]\end{aligned}$$

gives the proof. □ □

Now we define the faithful normal normalized trace τ_f on M_f and the trace preserving conditional expectation E_{N_f}.

Proposition 2.2. *For a nontrivial projection $f \in N$, we have the followings:*

(i) The faithful normal normalized trace τ_f on M_f is given by

$$\tau_f(fxf) = \tau(f)^{-1}\tau(fxf), \quad x \in M.$$

(ii) The unique τ_f-preserving conditional expectation E_{N_f} is given by

$$E_{N_f}(fxf) = E_N(fxf), \; x \in M.$$

Proof. (i) Since τ (resp. $\tau|_{M_f}$) is a faithful normal finite trace on M (resp. on M_f), τ_f is a scalar multiple of $\tau|_{M_f}$. Since $\tau_f(f \cdot 1 \cdot f) = \tau_f(f) = \tau(f)^{-1}\tau(f) = 1$, τ_f is a normalized trace and the uniqueness of normalized trace on II_1-factor M_f, τ_f is the faithful normal normalized trace on M_f.

(ii) Since E_N is the τ-preserving conditional expectation, we have

$$\begin{aligned}\tau_f(E_{N_f}(fxf)) &= \tau(f)^{-1}\tau(E_N(fxf))\\ &= \tau(f)^{-1}\tau(fxf) = \tau_f(fxf), \; x \in M.\end{aligned}$$

Thus E_{N_f} is the unique τ_f-preserving conditional expectation. □ □

Moreover, for any projection f_0 in N with $f_0 \le f$, we have $E_N(f_0) = E_{N_f}(f_0)$ and

$$\begin{aligned}\tau_f(fE_N(x)f) &= \tau_f(E_N(fxf)) = \tau(f)^{-1}\tau(E_N(fxf))\\ &= \tau(f)^{-1}\tau(fxf) = \tau_f(fxf), \; x \in M.\end{aligned}$$

For a nontrivial projection $f \in N$, let $(M_f)_1$ be the basic construction for $N_f \subset M_f$ and $(M_1)_f$ reduction for M_1. The reduction $(M_1)_f$ is a II_1-subfactor of M_1, containing M_f and has the faithful normal normalized trace $\tau_1|_{(M_1)_f}$, where τ_1 is the faithful normal normalized trace on M_1. The trace preserving conditional expectation E_{M_f} onto M_f is defined by $E_{M_f}(fxf) = E_M(fxf), x \in M_1$. So $(M_1)_f$ and $(M_f)_1$ are II_1-factors containing M_f, as a subfactor.

We investigate the relation between the reduction $(M_1)_f$ and the basic construction $(M_f)_1$. Here we prove that there exists an isomorphism in the sense of that in Proposition 1.2 in [9] between them which fixes M_f.

THEOREM 2.3. *If f is a nontrivial projection in N, then there exists an isomorphism ϕ of $(M_f)_1$ onto $(M_1)_f$ such that $\phi(x) = x$, $x \in M_f$ and $\phi(e_{N_f}) = fe_N f$.*

Proof. Since f is a nontrivial projection in N, by Proposition 2.1, we have $[M_f : N_f] = [M : N]$ and $[(M_1)_f : M_f] = [M_1 : M] = [M : N]$. Consider $fe_N f \in (M_1)_f, fe_N = e_N f$ implies that $fe_N f$ is a projection and $fe_N f \in N'$. Moreover we have

$$E_{M_f}(fe_N f) = fE_M(e_N)f = [(M_1)_f : M_f]^{-1} 1_{M_f}$$

Thus by Proposition 1.2 in [9], our proof is over. □ □

3. The basic construction for an induced factor and a reduced factor

We study Jones' index for an induced II_1-factor and a reduced II_1-factor. We construct the basic extension $(M_f)_1$ for $N_f \subset M_f$ and compare $(M_f)_1$ with $(M_1)_f$, where f is a nontrivial projection in $N' \cap M$. If e is a projection in M', then $\dim_{M_e}(eH) = \tau'(e) \dim_M(H)$. For a projection $f \in N' \cap M$, if we consider the reduction M_f and the induction N_f, then $N_f \subset M_f$ is a pair of II_1-factors.

Here, we also define the faithful normal normalized trace on M_f and the trace preserving conditional expectation E_{N_f}. Note that for $f \in N' \cap M$ and $x \in N$, we have

$$E_N(f) \cdot x = E_N(fx) = E_N(xf) = x \cdot E_N(f),$$

which gives $E_N(f) = \lambda \cdot 1$ for some scalar λ.
Since $\tau(f) = \tau(E_N(f)) = \lambda$, we have $E_N(f) = \tau(f) \cdot 1$.

PROPOSITION 3.1. *For a nontrivial projection $f \in N' \cap M$, we have the followings:*

(i) If we define τ_f by

$$\tau_f(fxf) = \tau(f)^{-1}\tau(fxf),\ x \in M,$$

then τ_f gives the faithful normal normalized trace on M_f.

(ii) If we define $E_{N_f} : M_f \to N_f$ by

$$E_{N_f}(fxf) = \tau(f)^{-1} f \cdot E_N(fxf) \cdot f,\ x \in M,$$

then E_{N_f} is the τ_f-preserving conditional expectation.

Proof. (i) Since τ (resp. $\tau|_{M_f}$) is a faithful normal finite trace on M (resp. on M_f), τ_f is a scalar multiple of $\tau|_{M_f}$. Since $\tau_f(f \cdot 1 \cdot f) = \tau_f(f) = \tau(f)^{-1}\tau(f) = 1$, τ_f is a normalized trace and the uniqueness of normalized trace on II_1-factor M_f, τ_f is the faithful normal normalized trace on M_f.

(ii) For any $fxf \in M_f$, we have

$$\begin{aligned}\tau_f(E_{N_f}(fxf)) &= \tau_f(\tau(f)^{-1} f E_N(fxf) f)\\ &= (\tau(f)^{-2})\tau(E_N(f)E_N(fxf)) = \tau_f(fxf).\square\end{aligned}$$

□

Corollary 3.2. *For a nontrivial projection $f \in N' \cap M$, we have the followings:*

(i) For $x \in N$, we have $\tau_f(fxf) = \tau(x)$.

(ii) For any projection f_0 in $N' \cap M$ with $f_0 \leq f$, we have

$$\|E_N(f_0)\| \leq \|E_{N_f}(f_0)\|.$$

Proof. (i) For $x \in N$, we have $\tau(fxf) = \tau(E_N(fx)) = \tau(E_N(f) \cdot x) = \tau(f) \cdot \tau(x)$.
It follows that $\tau_f(fxf) = \tau(x)$.

(ii)

$$\begin{aligned}E_{N_f}(f_0) &= E_{N_f}(ff_0f)\\ &= \tau(f)^{-1} f E_N(ff_0f) f = \tau(f)^{-1} f E_N(f_0) f.\end{aligned}$$

From the equalities of

$$E_N(fE_N(f_0)f) = E_N(E_N(f_0)f) = E_N(f_0)E_N(f) = E_N(f_0)\tau(f)$$

and from the fact of $\|E_N\| = 1$, we have

$$\|E_N(f_0)\| = \tau(f)^{-1}\|E_N(fE_N(f_0)f)\| \leq \tau(f)^{-1}\|fE_N(f_0)f\|.$$

Thus we obtain $\|E_N(f_0)\| \leq \|E_{N_f}(f_0)\|$. □ □

Consider a pair of II_1-factors $N_f \subset M_f$, $f \in N' \cap M$, a projection, with the unique faithful normal normalized trace τ_f and the τ_f-preserving conditional expectation $E_{N_f} : M_f \to N_f$. The local index of N at f is defined by $[M : N]_f = [M_f : N_f]$. By Lemma 2.2.1 in [4], the index at f and the global index are related by the formula $[M : N]_f = [M : N] \cdot \tau(f) \cdot \tau'(f)$, where τ' is the trace on N'. Now we are ready to study the basic construction $(M_f)_1$ for $N_f \subset M_f$.

When Jones' index $[M : N]$ is finite, $[M_f : N_f]$ is also finite. $L^2(M_f, \tau_f)$ is the Hilbert space of the GNS representation of M_f and M_f acts on $L^2(M_f, \tau_f)$ by left multiplication. The canonical conjugation on $L^2(M_f, \tau_f)$ is denoted by J_f and J_f acts on the dense subspace $M_f \subset L^2(M_f, \tau_f)$ by $J_f(fxf) = (fxf)^*$. E_{N_f} is the restriction to M_f of the orthogonal projection e_{N_f} of $L^2(M_f, \tau_f)$ onto $L^2(N_f, \tau_f)$, which is the closure in $L^2(M_f, \tau_f)$ of N_f.

The following properties are easy consequences of the definition and proofs are straightforward computations.

1. $e_{N_f} x e_{N_f} = E_{N_f}(x) e_{N_f}$, $x \in M_f$.
2. $x \in M_f$, $x \in N_f$ iff $e_{N_f} x = x e_{N_f}$.
3. $N'_f = (M'_f \cup \{e_{N_f}\})''$.
4. J_f commutes with e_{N_f}.

If $(M_f)_1 = (M_f \cup \{e_{N_f}\})''$ denotes the von Neumann algebra on $L^2(M_f, \tau_f)$, then $(M_f)_1 = J_f N'_f J_f$. This is called the basic construction for $N_f \subset M_f$.

5. $(M_f)_1$ is a factor iff N_f is a factor.
6. $(M_f)_1$ is finite iff N'_f is finite.

There exists a trace $(\tau_f)_1$ on $(M_f)_1$ such that $(\tau_f)_1|_{M_f} = \tau_f$ and $E_{M_f}(e_{N_f}) = \lambda \cdot 1_{M_f}$, where E_{M_f} is the $(\tau_f)_1$-preserving conditional expectation of $(M_f)_1$ onto M_f and $\lambda > 0$ is a scalar.

7. Jones' index $[M_f : N_f]$ is given by
$[M_f : N_f] = \dim_{N_f}(L^2(M_f, \tau_f)) = [M : N] \cdot \tau(f) \cdot \tau'(f)$.
8. $E_{M_f}(e_{N_f}) = [M_f : N_f]^{-1} \cdot f$.

Next, we show a relationship between the basic construction $(M_f)_1$ for $N_f \subset M_f$ and the reduction $(M_1)_f$, for a nontrivial projection f in $N' \cap M$.

THEOREM 3.3. *If f is a nontrivial projection in $N' \cap M$, then there exists no isomorphism between $(M_f)_1$ and $(M_1)_f$ which fixes M_f and sends e_{N_f} to $fe_N f$.*

Proof. Suppose that there exists an isomorphism between $(M_f)_1$ and $(M_1)_f$ which fixes M_f and sends e_{N_f} to fe_Nf.,where f is a nontrivial projection in $N' \cap M$. Then, by Proposition 1.2 in [9], $[(M_f)_1 : M_f] = [(M_1)_f : M_f]$ must be hold. But since f is a nontrivial projection in $N' \cap M$, we have

$$[(M_f)_1 : M_f] = [M_f : N_f] = [M : N] \cdot \tau(f) \cdot \tau'(f) \neq [M : N]$$

and by Proposition 2.1 we have $[(M_1)_f : M_f] = [M_1 : M] = [M : N]$. It follows that $[(M_f)_1 : M_f] \neq [(M_1)_f : M_f]$, which is a contradiction.□□

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