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ON THE KNOTTED ELASTIC CURVES

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ABSTRACT. According to the Bernoulli-Euler theory of elastic rods the bending energy of the wire is proportional to the total squared curvature of γ , which we will denote by $F(\gamma)=\int_\gamma k^2\,ds$. If the result of J.Langer and D.Singer [3] extend to knotted elastic curve, then we obtain the following. Let $\{\gamma, M\}$ be a closed knotted elastic curve. If the curvature of γ is nonzero for everywhere, then γ lies on torus.

I. Introduction

Elastic curve (or elastica) and its generalizations have long been of interest in the context of elasticity theory. The elastica as a purely geometrical entity seems to have been largely ignored (for historical references concerning the classical elastica, we refer to the recent survey by Truesdel [2]).

Elastic curve is a mathematical model of Peano curve. And elastic energy (bending energy) is critical for $\mathcal{T}$ defined on regular curves. Euler was able to obtain a good qualitative description of all plane elastic curves. In fact, Peano curve not only has a curve but also knot.Thus elastic curve is not complete mathematical model of Peano curve. Here, in order to establish a mathematical model, consider the energy which is the sum of elastic energy and knotted energy. And define the curve its energy is critical.

II. Main theorem

All curves, functions, vecterfields will be assumed C^∞ class. For 3-dimensional Euclidean space $\mathbf{R}^3$, Euclidean inner product will be denoted by $<\ ,\ >$ and the Euclidean norm by $|\ |$.

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Let $\gamma = \gamma(t) : [t_1, t_2] \to \mathbf{R}^3$ be a holomorphic C^∞-class curve. v will denote the velocity of γ and T, k will denote unite tangent vector and curvature of γ. (i.e, $v = |\frac{d\gamma}{dt}|, T = \frac{1}{v}\frac{d\gamma}{dt}, k = |\nabla_T T|$ where $\nabla_T = \frac{1}{v}\frac{d}{dt}$)

Define functional $\mathcal{F}$ by $\mathcal{F}(\gamma) = \int_{t_1}^{t_2} k^2 v dt$ and we called it elastic energy of γ.

Let M be a unit normal vector field along γ and $\{\gamma, M\}$ be a curve with unit normal vector field.

DEFINITION 1. Let $\{\gamma, M\}$ be a curve with unit normal vectorfield and its domain is $[t_1, t_2]$. Define a function h on $[t_1, t_2]$ by $h =< \nabla_T M, L >$, and we called it a knot function of $\{\gamma, M\}$. Here $L = T \times M$, $\times$ is exterior product in $\mathbf{R}^3$. $h(t)$ be a quantity of knot of M at $\gamma(t)$.

REMARK. If M is parallel to normal connection along to γ, then $h \equiv 0$.

DEFINITION 2. $\{\gamma, M\}, v, h$ are the same notation as above.

(1) $\int_{t_1}^{t_2} h^2 v dt$ is called a knotted energy of $\{\gamma, M\}$.

(2) Let $\epsilon > 0$ be a constant. Define a functional $\mathcal{T}_\epsilon$ with respect to curve with unit normal vector field by

$$\mathcal{T}_\epsilon(\{\gamma, M\}) = \mathcal{T}(\gamma) + \epsilon \int_{t_1}^{t_2} h^2 v dt.$$

$\mathcal{T}_\epsilon(\{\gamma, M\})$ is called knotted elastic energy of coefficient ϵ of $\{\gamma, M\}$. Here, domain of $\mathcal{T}_\epsilon$ is the set of all curve with unit normal vector field.

DEFINITION 3. Let $t_0 > 0, \phi \in \mathbf{R}/2\pi\mathbf{Z}$. $\{\gamma, M\}$ is called period t_0 if the following two conditions are satisfied.

(1) $\gamma(t + t_0) = \gamma(t)$.

(2) $M(t + t_0) = R(\phi)M(t)$ where $R(\phi) : T^2\mathbf{R}^3 \to T^\perp\mathbf{R}^3$ be a rotation of angle ϕ in each fiber of normal bundle $T^\perp\mathbf{R}^3$ along γ. (orientation is $R(\frac{\pi}{2})M(t) = L(t)$)

Let $C(t_0, \phi)$ be a set of all curves with unit normal vector field its period is t_0. Then we obtain a following lemma from the a first variation formula.

LEMMA 1. *$\{\gamma, M\} \in C(t_0, \phi)$ (its length $\int_0^{t_0} v dt$ is fixed) is critical point of $\mathcal{T}_\epsilon$ iff there exist real numbers μ, σ such that the following are satisfied.*

(1) $\nabla_T[2(\nabla_T)^2 T + (3k^2 - \mu + \epsilon h^2)T - 2\epsilon h R(\frac{\pi}{2})(\nabla_T T)] = 0.$
(2) $h(t) = \sigma.$

DEFINITION 4. Let $\{\gamma, M\}$ be a curve with unit normal vector field with velocity 1 (i.e. $v \equiv 1$). If (1) and (2) of the Lemma 1 are satisfied, then $\{\gamma, M\}$ is called a knotted elastic curve, σ is called a knot parameter of $\{\gamma, M\}$.

Let $l > 0, \phi \in \mathbf{R}/2\pi\mathbf{Z}$ and $\mathcal{U}C(l, \phi)$ be the set of velocity 1 of γ in $C(l, \phi)$ and $\{\gamma, M\}$ be an element of $\mathcal{U}C(l, \phi)$.

Define

$$T_{\{\gamma,M\}}C(l,\phi) = \left\{ (\Lambda, f) \mid \begin{array}{c} \Lambda \text{ is vector field of period} \\ l \text{ along } \gamma,\ f \text{ is a function} \\ \text{of period } l \end{array} \right\},$$

$$T_{\{\gamma,M\}}\mathcal{U}C(l,\phi) = \left\{ (\Lambda, f) \mid \begin{array}{c} (\Lambda, f) \ \in \ T\{\gamma, M\}C(l,\phi) \\ \text{i.e.} < \nabla_T \Lambda, T >= 0 \end{array} \right\}.$$

Then the following lemma is satisfied.

LEMMA 2. *For the variation $\{\gamma, M\}_\lambda$ of $\{\gamma, M\}$ in $\mathcal{U}C(l, \phi)$, $\{\gamma, M\}_\lambda = \{\gamma_\lambda, M_\lambda\}$ $(-\lambda_0 < \lambda < \lambda_0, \{\gamma, M\}_0 = \{\gamma, M\})$ $(\frac{\partial \gamma_l}{\partial \lambda}|_{\lambda=0}, < \frac{\partial M_\lambda}{\partial \lambda}, L_\lambda > |_{\lambda=0}) \in T_{\{\gamma,M\}}\mathcal{U}C(l,\phi)$. Left side is called a variational vector field of variation $\{\gamma, M\}_\lambda$*

Conversely, for any $(\Lambda, f) \in T_{\{\gamma,M\}}\mathcal{U}C(l,\phi)$, there exist a variation $\{\gamma, M\}_\lambda$ of $\{\gamma, M\}$ in $\mathcal{U}C(l,\phi)$ such that $\frac{\partial \gamma_l}{\partial \lambda}|_{\lambda=0} = \Lambda$, $< \frac{\partial M_\lambda}{\partial \lambda}, L_\lambda >$ $|_{\lambda=0} = f$.

In the above Lemma 2, $T_{\{\gamma,M\}}\mathcal{U}C(l,\phi)$ is tangent space of $\mathcal{U}C(l,\phi)$ at $\{\gamma, M\}$.

LEMMA 3. *Let $\{\gamma, M\}$ be a knotted elastic curve and also variational vector field of $\{\gamma, M\}_\lambda$. Then*

$$\frac{d^2}{d\lambda^2}|_{\lambda=0}\mathcal{T}_\epsilon(\{\gamma, M\}_\lambda) = \int_0^l < \mathcal{T}_{\{\gamma,M\}}(\Lambda, f), (\Lambda, f) > ds$$

where

$$\begin{aligned}\mathcal{T}_{\{\gamma,M\}}(\Lambda,f) = (p[\nabla_T\{2(\nabla_T)^3\Lambda + (3k^2-\mu+\epsilon\sigma^2)\nabla_T\Lambda \\ - 2\epsilon\sigma R(\frac{\pi}{2})((\nabla_T)^2\Lambda - <(\nabla_T)^2\Lambda, T> T) \\ - 2\epsilon(<\nabla_T\Lambda, R(\frac{\pi}{2})(\nabla_T T)> + Tf)R(\frac{\pi}{2})(\nabla_T T)\}] \\ - 2\epsilon(T<\nabla_T\Lambda, R(\frac{\pi}{2})(\nabla_T T)> + T^2 f))\end{aligned}$$

($p : T_{\{\gamma,M\}}C(l,\phi) \to T_{\{\gamma,M\}}\mathcal{U}C(l,\phi)$ is an orthogonal projection with respect to L^2-inner product)

We consider eigenvalue problem $\mathcal{T}_{\{\gamma,M\}}(\Lambda,f) = p(\Lambda,f), \phi \in \mathbf{R}$, we can obtain the following theorem.

THEOREM 4. *Let γ be a circle with radius 1 and $\{\gamma,M\}$ be a knotted elastic curve. Then the eigenvalue of Jacobi operator $\mathcal{T}_{\{\gamma,M\}}$ has the following properties.*

(1) *The eigenvalue of $\mathcal{T}_{\{\gamma,M\}}$ is positive whenever $0 \le \epsilon^2\sigma^2 < 3$.*
(2) *There exists a non-trivial eigenvector its eigen value 0 whenever $\epsilon^2\sigma^2 = m^2 - 1 (2 \le m,\ m$ is integer).*
(3) *There exists a negative eigenvalue whenever $\epsilon^2\sigma^2 > 3$.*

The curvature k and torsion τ of elastic curve are represented by elliptic function. By J. Langer and D. Singer [3], for every elastic curve γ in $\mathbf{R}^3$ there is naturally associated to γ a cylindrical coordinate system (r,θ,z) on $\mathbf{R}^3$, the restrictions to γ of the coordinate fields $\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta} \frac{\partial}{\partial z}$ being expressible in terms of k,τ,T,N,B. Thus we can see the following theorem.

THEOREM 5 [J. LANGER AND D. SINGER][3]. *Let γ be a closed elastic curve. Then γ lies on embedded tori of revolution.*

If the proof of the above theorem extend to a knotted elastic curve, then we obtain the following result.

THEOREM 6. *Let $\{\gamma,M\}$ be a closed knotted elastic curve. Suppose that, the curvature k of γ is nonzero for everywhere. Then γ lies on torus.*

PROOF. Let $\{\gamma, M\}$ be a knotted elastic curve. First of all, find the curvature k and torsion τ. Application of the Frenet formula for (1) of [Lemma 1] leads to the ordinary differential equation for k, τ. Solution of the equation is concretely represented by Jacobi function. In this representation, k and τ are periodic functions with the same period and if k is constant then τ is also constant. Secondly, we will construct a cylindrical coordinate system. If k is constant, then γ is a straight line or a circle. Suppose that k is not constant.

If $\gamma = \gamma(s)$ be a curve with velocity 1 and its curvature is positive at every point, Then Λ is a vector field along γ. Λ extends to a killing vector field on $\mathbf{R}^3$ iff Λ satisfies the following.

$$< \nabla_T \Lambda, T >= 0 \qquad (a)$$

$$< (\nabla_T)^2 \Lambda, N >= 0 \qquad (b)$$

$$< (\nabla_T)^3 \Lambda - \frac{k_s}{k} (\nabla_T)^2 \Lambda + k^2 \nabla_T \Lambda, B >= 0 \qquad (c)$$

where T, N, B form the Frenet frame for γ. Put

$$J_0 = 2(\nabla_T)^2 T + (3k^2 - \mu + \epsilon\sigma^2)T - 2\epsilon\sigma R(\frac{\pi}{2})(\nabla_T T)$$

$$H = \epsilon\sigma T + kB$$

$$J_1 = H - \frac{< J_0, H >}{|J_0|^2} J_0 \ \ (\mu, \sigma \text{ are constant in Lema 1}).$$

Then J_0, H, J_1 is also killing along γ. Let $\overline{J}_0, \overline{J}_1$ be the extension of J_0, J_1 on $\mathbf{R}^3$. In (1) of Lemma 1, we can see $\overline{J}_0$ is constant vector field. Thus $\overline{J}_1$ is a rotation field perpendicular to $\overline{J}_0$. By the above statement we obtain a cylindrical coordinate (r, θ, z). It satisfies $\frac{\partial}{\partial z} = \frac{1}{|\overline{J}_0|}\overline{J}_0$, $\frac{\partial}{\partial \theta} = c\overline{J}_1$ where c is a positive constant. Setting $\gamma(s) = (r(s), \theta(s), z(s))$ one then obtains

$$r(s) = c|J_1(s)|$$

$$\theta_s(s) = \frac{< T, \frac{\partial}{\partial \theta} >}{|\frac{\partial}{\partial \theta}|^2} = \frac{< T, J_1(s) >}{c|J_1(s)|^2}$$

$$z_s(s) = < T, \frac{1}{|J_0(s)|} J_0(s) >$$

where J_0, J_1 are vector fields along γ and components of T, N, B are represented by curvature and torsion of γ.

If $\{\gamma, M\}$ is periodic, then r,z are also periodic and curve of γ in rz-plane is a simple closed curve. Thus every closed knotted elastic curve lies on torus of revolution. Since curvature and torsion of knotted elastic curve are periodic function with some period, γ is periodic iff $\Delta z = 0$ (i.e. $\frac{\Delta\theta}{2\pi} \in \mathbf{Q}$) $\square$

THEOREM 7 [J.LANGER AND D.SINGER][3]. *For any closed elastic curve* $-\pi \leq \Delta\theta \leq 0, \frac{\Delta\theta}{2\pi} \in \mathbf{Q}$ *and conversely for any* ψ, *such that* $-\pi \leq \psi \leq 0, \frac{\psi}{2\pi} \in \mathbf{Q}$ *there exists a unique closed elastic curve such that* $\Delta\theta = \psi$.

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