Kangweon-Kyungki Math. Jour. **5** (1997), No. 1, pp. 55–60

APPROXIMATION OF RELIABILITY IMPORTANCE FOR CONTINUUM STRUCTURE FUNCTIONS

SeungMin Lee and RakJoong Kim

Abstract. A continuum structure function(CSF) is a non-decreasing mapping from the unit hypercube to the unit interval. The reliability importance of component i in a CSF at system level α, $R_i(\alpha)$ say, is zero if and only if component i is almost irrelevant to the system at level α. A condition to check whether a component is almost irrelevant to the system is presented. It is shown that $R_i^{(m)}(\alpha) \to R_i(\alpha)$ uniformly as $m \to \infty$ where each $R_i^{(m)}(\alpha)$ is readily calculated.

0. Introduction

Let $\phi : \{0,1\}^n \to \{0,1\}$ be a binary coherent structure function and let $h : [0,1]^n \to [0,1]$ be the corresponding reliability function. Birnbaum[6] defines the reliability importance of component i as

$$I(i) = \frac{\partial h(\hat{p})}{\partial p_i} = h(1_i, \hat{p}) - h(0_i, \hat{p}), \quad i = 1, 2, \cdots, n,$$

writing $(\beta_i, \hat{p}) = (p_1, p_2, \cdots, p_{i-1}, \beta, p_{i+1}, \cdots, p_n)$ where $p_i = P(X_i = 1)$ and where $X_1, X_2, \cdots, X_n$ are independent binary random variables denoting the states of the components of ϕ. This concept is extended by Kim and Baxter[9] to the continuum case.

Let Δ denote the unit hypercube $[0,1]^n$. A mapping $\gamma : \Delta \to [0,1]$ which is non-decreasing in each argument and which satisfies $\gamma(\hat{0}) = 0$

Received December 1, 1996.
1991 Mathematics Subject Classification: 60E05.
Key words and phrases: reliability importance, continuum structure function, component relevancy, lower extreme vector, stochastic performance function.

and $\gamma(\hat{1}) = 1$, writing $\hat{\alpha} = (\alpha, \alpha, \cdots, \alpha)$, is said to be a continuum structure function(CSF). Such functions are of interest in reliability theory e.g. Baxter[1],[2], where $x_1, x_2, \cdots, x_n$ denote the states of the component $C = \{1, 2, \cdots, n\}$ of system and $\gamma(\hat{x})$ denotes the state of the system. Let $U_\alpha = \{\hat{x} \in \Delta | \gamma(\hat{x}) \geq \alpha\}$, $0 \leq \alpha \leq 1$ and $\hat{\delta}_\alpha$ denote the intersection of ∂u_α, the boundary of U_α in Δ, and $\{\hat{\alpha} | 0 \leq \alpha \leq 1\}$, the diagonal of Δ. We say that $\hat{\delta}_\alpha$ is the key vector of U_α and we call δ_α the corresponding key element. Kim and Baxter[9] use the key element to define reliability importance when $\hat{X}$ is a random vector: they define the reliability importance of component i at level $\alpha \in (0, 1]$ as

$$R_i(\alpha) = P\{\gamma(\hat{X}) \geq \alpha | X_i \geq \delta_\alpha\} - P\{\gamma(\hat{X}) \geq \alpha | X_i < \delta_\alpha\},$$

$i = 1, 2, \cdots, n$. We note that the reliability importance $R_i(\alpha)$ of component i depends on the state α of the system. For any component i and any subset $A \subset \Delta$, we set $A^i = \{\hat{x} \in \Delta | (\cdot_i, \hat{x}) = (\cdot_i, \hat{z})$ for some $\hat{z} \in A\}$. Notice that $A \subset A^i$ and that $A = A^i$ if and only if whether or not $\hat{x} \in A$ does not depend on the state of component i. Baxter and Lee[4] defines that component i is almost irrelevant to γ if there exists a subset $E_\alpha \subset \Delta$ such that , for any $\alpha \in [0, 1]$,

$$\mu(E_\alpha^c) = 0 \quad \text{and} \quad U_\alpha \cap E_\alpha = (U_\alpha \cap E_\alpha)^i \cap E_\alpha$$

where μ denotes Lebesgue measure on $\mathcal{R}^n$ and also show that, under some conditions, $R_i(\alpha) = 0$ if and only if component i is almost irrelevant to γ.

1. Component Relevancy to the System

The CSF γ is weakly coherent if and only if $\sup_{\hat{x} \in \Delta}[\gamma(1_i, \hat{x}) - \gamma(0_i, \hat{x})] > 0$ for $i = 1, 2, \cdots, n$; this is the weakest form of component relevancy[2]. It is reasonable to say component i to be irrelevant(or inessential) if $\gamma(1_i, \hat{x}) - \gamma(0_i, \hat{x}) = 0$ for all $\hat{x} \in \Delta$. If equality holds for all $\hat{x} \in \Delta - A, \mu(A) = 0$, then it is reasonable to define component i to be irrelevant a.e. (almost irrelevant) to the system.

A subset $U \subset \Delta$ is said to be an upper set if $\hat{y} \in \Delta$ whenever $\hat{y} \geq \hat{x}$ and $\hat{x} \in U$. If U is an upper set, the vector $\hat{y}$ is said to be a lower extreme vector of U if $\{\hat{x} \in \Delta | \hat{x} \leq \hat{y}\} \cap \bar{U} = \{\hat{y}\}$, where $\bar{U}$ denotes the closure of U_α in Δ. For any CSF γ, let Q_α denote the set of lower extreme vectors of U_α; notice that $Q_\alpha \subset \partial U_\alpha$ and that if U_α is closed, then $Q_\alpha = P_\alpha$ [4].

PROPOSITION 1.1. *The component i in a CSF γ is almost irrelevant to γ if and only if* $\sup_{\hat{x}\in\Delta}[\gamma(1_i, \hat{x}) - \gamma(0_i, \hat{x})] = 0$ *a.e.* $[\mu]$.

Proof. ("if") Suppose that $\sup_{\hat{x}\in\Delta}[\gamma(1_i, \hat{x}) - \gamma(0_i, \hat{x})] = 0$ a.e. $[\mu]$. Define the function $\gamma' : \Delta \to [0, 1]$ by $\gamma'(0_i, \hat{0}) = 0$, $\gamma'(1_i, \hat{1}) = 1$ and $\gamma'(x_i, \hat{x}) = \gamma(0_i, \hat{x})$ for all $\hat{x}$ such that $\hat{x} \neq \hat{1}$ or $\hat{x} \neq \hat{0}$. Define $E = \{\hat{x} \in \Delta | \gamma'(\hat{x}) = \gamma(\hat{x})\}$; clearly $\mu(E^c) = 0$. Choose $\alpha \in [0, 1]$ and define $V_\alpha = \{\hat{x} \in \Delta | \gamma'(\hat{x}) \geq \alpha\}$. Then $U_\alpha \cap E = V_\alpha \cap E$ and, since $\sup_{\hat{x}\in\Delta}[\gamma'(1_i, \hat{x}) - \gamma'(0_i, \hat{x})] = 0$, $V_\alpha^i = V_\alpha$. Choose $\hat{x} \in (U_\alpha \cap E)^i \cap E$. Since $(U_\alpha \cap E)^i = (V_\alpha \cap E) \subset V_\alpha^i = V_\alpha$, $\hat{x} \in V_\alpha \cap E$, so $\hat{x} \in U_\alpha \cap E$. Thus $(U_\alpha \cap E)^i \cap E \subset U_\alpha \cap E$. Since the reverse inclusion always holds, we have shown that $(U_\alpha \cap E)^i \cap E = U_\alpha \cap E$ where $\mu(E) = 0$. Since α is arbitrary, component i is almost irrelevant to γ as claimed.

("only if") Suppose that component i is almost irrelevant to the system γ. Then, by Proposition 2.1 of Baxter and Lee[4], $y_i = 0$ for all $\hat{y} \in Q_\alpha \cap [0, 1]^n$. Define $V_\alpha = \cup_{\hat{y}\in Q_\alpha\cap[0,1]^n} U(\hat{y})$ for each $\alpha \in [0, 1]$ where $U(\hat{y}) = \{\hat{x} \in \Delta | \hat{x} \geq \hat{y}\}$, and define the function $\gamma' : \Delta \to [0, 1]$ by $\gamma'(\cdot_i, \hat{0}) = 0, \gamma'(\cdot_i, \hat{1}) = 1$ and $\gamma'(\hat{x}) \geq \alpha$ if and only if $\hat{x} \in V_\alpha - \{\hat{x} \in \Delta | x_j = 0, j \neq 1\} (0 \leq \alpha \leq 1)$. Define $W_\alpha = \{\hat{x} \in \Delta | \gamma'(\hat{x}) \geq \alpha\}$ and observe that $W_0 = V_0 = \Delta$ and that $\gamma'(\hat{0}) = 0$ and $\gamma'(\hat{1}) = 1$. It suffices to show that $\gamma' = \gamma$ a.e. $[\mu]$ and $\sup_{\hat{x}\in\Delta}[\gamma'(1_i, \hat{x}) - \gamma'(0_i, \hat{x})] = 0$. Firstly, we show that $\gamma' = \gamma$ a.e. $[\mu]$. Define $E = (0, 1)^n - D_\gamma$ where D_γ is the set of all discontinuity points of γ. We claim that $U_\alpha \cap E = V_\alpha \cap E$ for all $\alpha \in [0, 1]$. If $\alpha = 0$, the result is trivial, so choose $\alpha \in (0, 1]$ and $\hat{x} \in U_\alpha \cap E$. Since $\hat{x} \in U_\alpha$, there exists a vector $\hat{y} \in Q_\alpha$ such that $\hat{y} \leq \hat{x}$; since $\hat{x} \in E, x_j < 1$ for $j = 1, 2, \cdots, n$. Thus $y_j \leq x_j < 1$, $j = 1, 2, \cdots, n$, so $\hat{y} \in Q_\alpha \cap [0, 1]^n$, and hence $\hat{x} \in V_\alpha$. Since $\hat{x} \in E$, we have shown that $U_\alpha \cap E \subset V_\alpha \cap E$. Now choose $\hat{z} \in V_\alpha \cap E$. Then $\hat{z} \geq \hat{y}$ for some $\hat{y} \in Q_\alpha \cap [0, 1]^n$ by definition of V_α, and hence $\hat{z} \in \bar{U}_\alpha$. If $\hat{z} \in U_\alpha$, i.e., $\hat{z} \in \bar{U}_\alpha - U_\alpha$, then $\hat{x} \in \bar{U}_\alpha$, $\gamma(\hat{x}+) \geq \alpha$ whereas $\hat{x} \notin U_\alpha$ so that $\gamma(\hat{x}) < \alpha$, and hence $\hat{x} \in D_\gamma$, contradicting the assumption that $\hat{x} \in E \cap D_\gamma^c$. Thus $V_\alpha \cap E \subset U_\alpha \cap E$, and hence $U_\alpha \cap E = V_\alpha \cap E$ for all $\alpha \in [0, 1]$ as claimed. Further, since

$V_\alpha \cap E = W_\alpha \cap E$, $\gamma'(\hat{x}) = \gamma(\hat{x})$ for all $\hat{x} \in E$. But $E^c \subset D_\gamma$ and $\mu(D_\gamma) = 0$ by Lemma 2.2 of Baxter and Lee[5], so $\mu(E^c) = 0$. Thus $\gamma' = \gamma$ a.e. $[\mu]$. Secondly, we show that $\sup_{\hat{x}\in\Delta}[\gamma'(1_i, \hat{x}) - \gamma'(0_i, \hat{x})] = 0$. It suffices to show that $V_\alpha^i = V_\alpha$ for all $\alpha \in [0, 1]$. Since $V_0^i = V_0 = \Delta$, choose $\alpha > 0$ and $\hat{x} \in V_\alpha^i$, i.e., $(\cdot_i, \hat{x}) = (\cdot_i, \hat{z})$ for some $\hat{z} \in V_\alpha$. Since $\hat{z} \in V_\alpha$, $\hat{z} \geq \hat{y}$ for some $\hat{y} \in Q_\alpha \cap [0, 1]^n$ and since, by assumption, component i is almost irrelevant to γ, $y_i = 0$ by Proposition 2.1 of Baxter and Lee[4]. Then $x_i \geq 0 = y_i$ and, since $\hat{z} \geq \hat{y}, x_j = z_j \geq y_j$ for $j \neq 1$ so $\hat{x} \geq \hat{y}$ and hence $\hat{x} \in V_\alpha$. Thus $V_\alpha^i \subset V_\alpha$ and, since the reverse inclusion always holds, $V_\alpha^i = V_\alpha$ as claimed. This completes the proof. □ □

2. Approximation of the reliability importance

Suppose that $X_1, X_2, \cdots, X_n$, the states of the components, are independent random variables defined on the same probability space $(\Omega, \mathcal{F}, P)$ and that γ is right-continuous so that $\gamma(\hat{X})$ is $\mathcal{F}$-measurable. A computationally tractable expression for the stochastic performance function $\Phi(\alpha) = P\{\gamma(\hat{X}) \geq \alpha\}$ occurs only in certain special case and, although bounds can be constructed, these may not be appreciably easier to calculate than Φ itself. However, $\Phi(\alpha)$ can be easily evaluated if P_α is finite. We observe that if U_α is closed and P_α is finite, then

$$P\{\gamma(\hat{X}) \geq \alpha\} = \sum_{j=1}^{N}\prod_{i=1}^{n}\bar{F}_i(y_i^{(j)}) - \sum\sum_{j_1<j_2}\prod_{i=1}^{n}\bar{F}_i(y_i^{(j_1)} \vee y_i^{(j_2)})$$
$$+ \cdots + (-1)^{N-1}\prod_{i=1}^{n}\bar{F}_i(\max_{1\leq j\leq N} y_i^{(j)}),$$

writing $P_\alpha = \{\hat{y}^{(1)}, \cdots, \hat{y}^{(N)}\}$, the set of N minimal vectors, and $\bar{F}_i(x) = P\{X_i \geq x\}$, $i = 1, 2, \cdots, n$, i.e., that $P\{\gamma(\hat{X}) \geq \alpha\}$ is easily evaluated. Suppose that γ is right-continuous at $\hat{1}$ and define the mapping $\gamma' : \Delta \to [0, 1]$ by $\gamma'(\hat{X}) \geq \alpha$ if and only if $\hat{X} \in U_\alpha \cap D_{\alpha i}$ where $D_{\alpha i} = \{\hat{X} \in \Delta | X_i > \delta_\alpha\}$. Clearly γ' is a right-continuous CSF. Let $\Phi'(\alpha) = P\{\gamma'(\hat{X}) \geq \alpha\}$. Then $R_i(\alpha) = \Phi'(\alpha)/\bar{F}_i(\delta_\alpha) - [\Phi(\alpha) - \Phi'(\alpha)]/[1 - \bar{F}_i(\delta_\alpha)]$. We note that if P_α is finite, then $\Phi(\alpha)$ and $\Phi'(\alpha)$

are easily evaluated, and hence so is $R_i(\alpha)$. A CSF γ is called strongly increasing if $\gamma(\hat{x}) > \gamma(\hat{y})$ whenever $x_i > y_i$ for $i = 1, 2, \cdots, n$.

PROPOSITION 2.1. *Let γ be a strongly increasing CSF which is continuous at $\hat{0}$ and $\hat{1}$ and suppose that $X_1, \cdots, X_n$ are independent, absolutely continuous random variables, the support of each of which is the unit interval. Then there exists a sequence $\{\gamma_m\}$ of right-continuous CSF's for which each P_α is finite such that $R_i^{(m)} \to R_i$ uniformly as $m \to \infty$ on $[a, b]$, $0 < a < b < 1$, where $R_i^{(m)}$ is the reliability importance of γ_m.*

Proof. Define the mappings $\gamma'(\gamma'') : \Delta \to [0, 1]$ by $\gamma'(\hat{x})(\gamma''(\hat{x})) \geq \alpha$ if and only if $\hat{x} \in \bar{U}_\alpha(\hat{x} \in \bar{U}_\alpha \cap \bar{D}_{\alpha i})$. Clearly γ' and γ'' are right-continuous CSF's. Let

$$\Phi'(\alpha) = P\{\gamma'(\hat{X}) \geq \alpha\} \text{ and } \Phi''(\alpha) = P\{\gamma''(\hat{X}) \geq \alpha\}.$$

Then

$$R_i'(\alpha) = \Phi'(\alpha)/\bar{F}(\delta_\alpha) - [\Phi'(\alpha) - \Phi''(\alpha)]/[1 - \bar{F}_i(\delta_\alpha)]$$

where $R_i'(\alpha)$ is the reliability importance of γ'. Since X_i's are absolutely continuous and $\mu(\partial U_\alpha) = 0$ and $\mu(\partial D_{\alpha_i}) = 0$ by Lemma 2.1 of Baxter and Lee[5], $R_i'(\alpha) = R_i(\alpha)$, $0 < \alpha < 1$. Since γ', γ'' are right-continuous, there exist sequences $\{\gamma_m'\}, \{\gamma_m''\}$ of right-continuous CSF's, the P_α's of which are all finite, such that $\Phi_m' \to \Phi'$ and $\Phi_m'' \to \Phi''$ pointwise as $m \to \infty$ where $\Phi_m'(\alpha)$ and $\Phi_m''(\alpha)$ are $P\{\gamma_m'(\hat{X}) \geq \alpha\}$ and $P\{\gamma_m''(\hat{X}) \geq \alpha\}$ respectively. Then $R_i^{'(m)}(\alpha) \to R_i'(\alpha)$ pointwise as $m \to \infty$, and hence $R_i^{'(m)}(\alpha) \to R_i(\alpha)$ pointwise as $m \to \infty$. Since each X_i has support $[0, 1]$ and γ is continuous at $\hat{0}$ and $\hat{1}$, we have $0 < P\{X_i \geq \delta_\alpha\} < 1$ for $0 < \alpha < 1$ and $i = 1, 2, \cdots, n$. Further, since γ is strongly increasing, each of the terms $\Phi'(\alpha), \Phi''(\alpha), P_{\hat{X}}(D_{\alpha i})$ is a continuous function of α by the argument similar to the proof Theorem 3.1 of Baxter and Lee[4], and hence $R_i^{'(m)} \to R_i$ uniformly on $[a, b]$, $0 < a < b < 1$. □ □

REMARK. The approximation procedure above is only practicable for small or moderate values of n since the computational complexity of the calculation grows rapidly with n.

References

1. Baxter, L.A., *Continuum Structures I*, Journal of Applied Probability **21** (1984), 802-815.
2. Baxter, L.A., *Continuum Structures II*, Mathematical Proceedings of Cambridge Philosophical Society **99** (1986), 331-338.
3. Baxter, L.A. and Lee, S.M., *Structure Functions with Finite Minimal Vector Sets*, Journal of Applied Probability **26** (1989), 196-201.
4. Baxter, L.A. and Lee, S.M., *Further Properties of Reliability Importance for Continuum Structure Functions*, Probability in the Engineering and Informational Sciences **3** (1989), 237-246.
5. Baxter, L.A. and Lee, S.M., *A weak Convergence Theorem for Continuum Structure Functions*, Journal of Mathematical Analysis and Applications **148(2)** (1990), 463-468.
6. Birnbaum, Z.W., *"On the Importance of Difference Components in a Multicomponent System" in Multivariate Analysis-II, ed. P. R. Krishnaiah, New York* (1969), Academic Press, 581-592.
7. Block, H.W and Savits, T.H, *A Decomposition for Multistate Monotone Systems*, Journal of Applied Probability **19** (1982), 391-402.
8. Block, H.W and Savits, T.H, *Continuous Multistate Structure Functions*, Operations Research **32** (1984), 703-714.
9. Kim, C. and Baxter, L.A., *Reliability Importance for Continuum Structure Functions*, Journal of Applied Probability **24** (1987), 779-785.

SeungMin Lee
Department of Statistics
Hallym University
Chunchon 200-702, Korea

RakJoong Kim
Department of Mathematics
Hallym University
Chunchon 200-702, Korea