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GENERALIZED REIDEMEISTER NUMBER ON A TRANSFORMATION GROUP

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Abstract. In this paper we study the generalized Reidemeister number $R(\varphi, \psi)$ for a self-map $(\varphi, \psi) : (X, G) \to (X, G)$ of a transformation group (X, G), as an extension of the Reidemeister number $R(f)$ for a self-map $f : X \to X$ of a topological space X.

1. Introduction

It is observed that the number of the fixed point classes for a self-map $f : X \to X$ of a compact connected ANR could be calculated by defining an equivalence relation on the fundamental group $\pi_1(X, x_0)$.

The number of equivalence classes of $\pi_1(X, x_0)$, the Reidemeister number $R(f)$, equals the number of the fixed point classes of f.

F.Rhodes [3] represented the fundamental group $\sigma(X, x_0, G)$ of a transformation group (X, G), a group G of homeomorphisms of a space X, as a generalization of the fundamental group $\pi_1(X, x_0)$ of a topological space X.

In the present paper we defined the generalized Reidemeister number $R(\varphi, \psi)$ for a self-map (φ, ψ): $(X, G) \to (X, G)$ of the transformation group (X, G) and investigate its homotopy invariance. We also give the algebraic estimation of the definition of $R(\varphi, \psi)$ in the same way as in [2].

2. Preliminaries

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In this paper, a transformation group is a pair (X, G), where X is a path connected space with base point x_0 and G is a group of homeomorphisms of X. A map (φ, ψ): $(X, G) \to (X, G)$ consists of a continuous map $\varphi : X \to X$ and a homomorphism $\psi : G \to G$ such that $\varphi(g\, x) = \psi(g)\varphi(x)$ for every pair (x, g).

Given any element g of G, a path α of order g with base point x_0 is a continuous map $\alpha : I \to X$ such that $\alpha(0) = x_0$ and $\alpha(1) = gx_0$. A path α of order g_1 and a path β of order g_2 form a new path $\alpha + g_1\beta$ of order g_1g_2 defined by the following equations

$$(\alpha + g_1\beta)(t) = \begin{cases} \alpha(2t), & 0 \le s \le \frac{1}{2}, \\ g_1\beta(2t-1), & \frac{1}{2} \le s \le 1. \end{cases}$$

Two paths α and β of the same order g are said to be homotopic if there is a continuous map $F : I \times I \to X$ such that

$$\begin{aligned} F(t, 0) &= \alpha(t), & 0 \le t \le 1, \\ F(t, 1) &= \beta(t), & 0 \le t \le 1, \\ F(0, s) &= x_0, & 0 \le s \le 1, \\ F(1, s) &= gx_0, & 0 \le s \le 1. \end{aligned}$$

The equivalence relation $\alpha \sim \beta$ denotes that α and β are homotopic paths of the same order. Denote the equivalence class containing a path α of order g by $[\alpha; g]$. Two homotopic classes of paths of different orders g_1 and g_2 are distinct, even if $g_1x_0 = g_2x_0$. F.Rhodes [3] showed that the set of homotopy classes of paths of prescribed order with the rule of composition * is a group, where * is defined by $[\alpha; g_1] * [\beta; g_2] = [\alpha + g_1\beta; g_1g_2]$. This group was called the fundamental group of (X, G) with base points x_0, and was denoted by $\sigma(X, x_0, G)$. He also proved that $\sigma(X, x_0, G)$ is an invariant of the base point x_0.

3. Main results

Let $(\varphi, \psi) : (X, G) \to (X, G)$ be a mapping. It is easy to see that if α is a path in X of order g with base point x_0 then $\varphi\alpha$ is a path in X of order $\psi(g)$ with base point $\varphi(x_0)$. Furthermore, if $\alpha \sim \beta$ then $\varphi\alpha \sim \varphi\beta$. Thus (φ, ψ) induces a homomorphism $(\varphi, \psi)_*$: $\sigma(X, x_0, G) \to \sigma(X, \varphi(x_0), G)$ defined by $(\varphi, \psi)_*[\alpha; g] = [\varphi\alpha; \psi(g)]$.

If λ is a path from $\varphi(x_0)$ to x_0, then λ induces an isomorphism

$$\lambda_* : \sigma(X, \varphi(x_0), G) \to \sigma(X, x_0, G)$$

defined by $\lambda_*[\alpha;\, g] = [\lambda\rho + \alpha + g\lambda;\, g]$ for each $[\alpha; g] \in \sigma(X,\, \varphi(x_0), G)$, where $\rho(t) = 1-t$. This isomorphism λ_* depends only on the homotopy class of λ.

Consider the composition

$$\sigma(X,\, x_0,\, G) \xrightarrow{(\varphi,\psi)_*} \sigma(X,\, \varphi(x_0),\, G) \xrightarrow{\lambda_*} \sigma(X,\, x_0,\, G).$$

Definition 3.1. Let $\lambda_*(\varphi,\, \psi)_* = (\varphi,\, \psi)_\sigma$. Two elements $[\alpha;\, g_1]$ and $[\beta;\, g_2]$ in $\sigma(X,\, x_0,\, G)$ are said to be $(\varphi,\, \psi)_\sigma-$*equivalent*, denoted by $[\alpha;\, g_1] \overset{(\varphi,\psi)_\sigma}{\sim} [\beta;\, g_2]$, if there exists $[\gamma; g] \in \sigma(X,\, x_0,\, G)$ such that $[\alpha;\, g_1] = [\gamma;\, g][\beta;\, g_2](\varphi,\, \psi)_\sigma([\gamma;\, g]^{-1})$. This is an equivalence relation on $\sigma(X,\, x_0,\, G)$. Let $\sigma(X,\, x_0,\, G)^{'}(\varphi,\, \psi)_\sigma$ be the set of equivalence classes of $\sigma(X,\, x_0, G)$ under $(\varphi,\, \psi)_\sigma$-equivalence.

The cardinality of $\sigma(X,\, x_0,\, G)^{'}(\varphi,\, \psi)_\sigma$ is the *algebraic Reidemeister number* of $(\varphi,\, \psi)_\sigma$, and is denoted by $R_*(\varphi,\, \psi)_\sigma$. With this view, we may define the *Reidemeister number* of a map $(\varphi,\, \psi);\, (X,\, G) \to (X,\, G)$, $R(\varphi,\, \psi)$, to be the algebraic Reidemeister number of $(\varphi,\, \psi)_\sigma$. In symbols,

$$R(\varphi,\, \psi) = R_*(\varphi,\, \psi)_\sigma = \#\sigma(X,\, x_0,\, G)^{'}(\varphi,\, \psi)_\sigma.$$

Lemma 3.2. *The definition of $R(\varphi,\, \psi)$ is independent of the choice of the path λ from $\varphi(x_0)$ to x_0.*

Proof. Let τ denote another path from $\varphi(x_0)$ to x_0. Then $\lambda^{-1}\tau$ is a loop at x_0 and therefore induces an inner automorphism

$$(\lambda^{-1}\tau)_* :\, \sigma(X,\, x_0,\, G) \to \sigma(X,\, x_0,\, G)$$

generated by the element $[\lambda^{-1}\tau;\, e]$, since

$$(\lambda^{-1}\tau)_*[\alpha;\, g] = [\lambda^{-1}\tau\rho;\, e][\alpha;\, g][\lambda^{-1}\tau;\, e].$$

Applying this automorphism to the left-hand side of $\lambda_*(\varphi,\,\psi)_*$ we have

$$R_*(\lambda_*(\varphi,\,\psi)_*) = R_*(\tau_*\lambda_*^{-1}\lambda_*(\varphi,\,\psi)_*) = R_*(\tau_*(\varphi,\,\psi)_*).$$

Hence we have independence of the path λ. □ □

For a given homotopy $F:\varphi_1 \cong \varphi_2 : X \to X$ and a given path $c: I \to X$, define the (diagonal) path $< F,\, c >: I \to X$ by $< F,\, c >(t) = F(c(t),\, t),\ 0 \le t \le 1$. Then the path $< F,\, c >$ preserves inverse in the following sense.

Lemma 3.3. [1] $< F,\, c >^{-1} = < F^{-1},\, c^{-1} >$.

Our first result is the following.

Theorem 3.4. (Homotopy Invariance) *Let* $(\varphi_1,\,\psi_1),\ (\varphi_2,\,\psi_2)$ *be self-maps of* $(X,\,G)$. *If* $F:\varphi_1 \cong \varphi_2 : X \to X$ *is homotopy from* φ_1 *to* φ_2, *then* $R(\varphi_1,\,\psi_1) = R(\varphi_2,\,\psi_2)$.

Proof. Let $x_0 \in X$. Then $< F,\, x_0 >$ is a path from $\varphi_1(x_0)$ to $\varphi_2(x_0)$. Thus the path $< F,\, x_0 >$ induces a homomorphism

$$< F,\, x_0 >_*: \sigma(X,\,\varphi_1(x_0),\,G) \to \sigma(X,\,\varphi_2(x_0),\,G).$$

So we obtain the following induced commutative diagram

$$\begin{array}{ccc} \sigma(X,x_0,G) & \xrightarrow{(\varphi_1,\psi_1)_*} & \sigma(X,\varphi_1(x_0),G) \\ (\varphi_2,\psi_2)_* \searrow & & \nearrow < F^{-1},x_0 >_* \\ & \sigma(X,\varphi_2(x_0),G) & \end{array}$$

From Lemma 3.2 and Lemma 3.3, we have

$$\begin{aligned} R(\varphi_1,\,\psi_1) =& R_*(\lambda_*(\varphi_1,\,\psi_1)_*) \\ =& R_*(\lambda_* < F,\, x_0 >_*^{-1} (\varphi_,\,\psi_2)_*) \\ =& R_*((< F^{-1},\, x_0 > \lambda)_*(\varphi_2,\,\psi_2)_*) \\ =& R(\varphi_2,\,\psi_2). \end{aligned}$$

Hence we complete the proof of theorem. □ □

Let $\sigma(X,\,x_0,\,G)'$ be a commutator subgroup of $\sigma(X,\,x_0,\,G)$ generated by the set

$$\{[\alpha;g_1][\beta;g_2][\alpha;g_1]]^{-1}[\beta;g_2]^{-1} \mid [\alpha;g_1][\beta;g_2] \in \sigma(X,\,x_0,\,G)\}.$$

For a convenient notation, we shall write $\bar{\sigma}(X,\,x_0,\,G)$ for the quotient group $\sigma(X,\,x_0,\,G)/\sigma(X,\,x_0,\,G)'$.

Theorem 3.5. *If* $(\varphi, \psi) : (X, G) \to (X, G)$ *is a self-map, then*

$$R(\varphi, \psi) \geq \# \operatorname{Coker}(1 - (\varphi, \psi))_{\bar{\sigma}} \geq 1,$$

where 1 and $(\varphi, \psi)_{\bar{\sigma}}$ *denote respectively the identity isomorphism and the endomorphism of* $\bar{\sigma}(X, x_0, G)$ *induced by* (φ, ψ).

Proof. Obviously, there exists a canonical homomorphism

$$\theta_\sigma : \sigma(X, x_0, G) \to \bar{\sigma}(X, x_0, G)$$

such that $\operatorname{Ker}\theta_\sigma = \sigma(X, x_0, G)'$. Hence the following diagram is commutative:

$$\begin{array}{ccc} \sigma(X, x_0, G) & \xrightarrow{(\varphi,\psi)_\sigma} & \sigma(X, x_0, G) \\ \downarrow \theta_\sigma & & \downarrow \theta_\sigma \\ \bar{\sigma}(X, x_0, G) & \xrightarrow{(\varphi,\psi)_{\bar{\sigma}}} & \bar{\sigma}(X, x_0, G) \end{array}$$

For $[\gamma; g] \in \sigma(X, x_0, G)$, any element of the $(\varphi, \psi)_\sigma$-equivalent class containing $[\beta; g_2]$ may be expressed in the form

$$[\alpha; g_1] = [\gamma; g][\beta; g_2](\varphi, \psi)_\sigma([\gamma; g]^{-1}).$$

From the above diagram, we can easily obtain

$$\begin{aligned} \theta_\sigma([\alpha; g_1]) &= \theta_\sigma([\gamma; g][\beta; g_2](\varphi, \psi)_\sigma([\gamma; g]^{-1})) \\ &= \theta_\sigma([\gamma; g]) + \theta_\sigma([\beta; g_2]) - \theta_\sigma(\varphi, \psi)_\sigma([\gamma; g]) \\ &= \theta_\sigma([\gamma; g]) + \theta_\sigma([\beta; g_2]) - (\varphi, \psi)_{\bar{\sigma}}\theta_\sigma([\gamma; g]) \\ &= \theta_\sigma([\beta; g_2]) + (1 - (\varphi, \psi)_{\bar{\sigma}})(\theta_\sigma([\gamma; g])). \end{aligned}$$

Thus there exists an element $[\gamma; g] \in \sigma(X, x_0, G)$ such that

$$\theta_\sigma([\alpha; g_1]) - \theta_\sigma([\beta; g_2]) = \\ (1 - (\varphi, \psi)_{\bar{\sigma}})(\theta_\sigma([\gamma; g])) \in (1 - (\varphi, \psi)_{\bar{\sigma}})(\bar{\sigma}(X, x_0, G)).$$

Let $\eta_{\bar{\sigma}} : \bar{\sigma}(X, x_0, G) \to \operatorname{Coker}(1 - (\varphi, \psi)_{\bar{\sigma}})$ be the natural projection. Now consider

$$\sigma(X, x_0, G) \xrightarrow{\theta_\sigma} \bar{\sigma}(X, x_0, G) \xrightarrow{\eta_\sigma} \operatorname{Coker}(1 - (\varphi, \psi)_{\bar{\sigma}}).$$

Since both θ_σ and $\eta_{\bar{\sigma}}$ are epimorphisms, $\eta_{\bar{\sigma}}\theta_\sigma$ is also an epimorphism. Moreover, the $\eta_{\bar{\sigma}}\theta_\sigma$ images of all element of a $(\varphi, \psi)_\sigma$-equivalent class are the same element of $\operatorname{Coker}(1 - (\varphi, \psi)_{\bar{\sigma}})$. This completes the proof of theorem. □ □

COROLLARY 3.6. *Let* $(\varphi, \psi) : (X, G) \to (X, G)$ *be a self-map. If* $\sigma(X, x_0, G)$ *is abelian and* G *is abelian, then*

$$R(\varphi, \psi) = \#\,\mathrm{Coker}(1 - (\varphi, \psi)_{\bar\sigma}).$$

Proof. If $\sigma(X, x_0, G)$ is abelian, then the natural homomorphism $\theta_\sigma : \sigma(X, x_0, G) \to \bar\sigma(X, x_0, G)$ is an isomorphism. From Definition 3.1, $[\alpha; g_1] \sim [\beta; g_2]$ if and only if there exists $[\gamma; g] \in \sigma(X, x_0, G)$ such that

$$\theta_\sigma([\alpha; g_1] - \theta_\sigma([\beta, g_2]) = (1 - (\varphi, \psi)_{\bar\sigma})(\theta_\sigma([\gamma; g])),$$

In other words,

$$\eta_{\bar\sigma}\theta_\sigma([\alpha; g_1]) - \eta_{\bar\sigma}\theta_\sigma([\beta; g_2]) = 0.$$

This completes the proof of theorem. □ □

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