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POSITIVE SOLUTIONS ON NONLINEAR BIHARMONIC EQUATION

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Abstract. In this paper we investigate the existence of positive solutions of a nonlinear biharmonic equation under Dirichlet boundary condition in a bounded open set Ω in $\mathbf{R}^n$, i.e.,

$$\begin{aligned} &\Delta^2 u + c\Delta u = bu^+ + s \qquad \text{in } \Omega, \\ &u = 0, \Delta u = 0 \qquad \text{on } \partial\Omega. \end{aligned}$$

0. Introduction

Let Ω be a bounded open set in $\mathbf{R}^n$ with smooth boundary $\partial\Omega$. In this paper, we shall concern with the nonlinear biharmonic problem

$$\begin{aligned} &\Delta^2 u + c\Delta u = bu^+ + s \qquad \text{in } \Omega, \\ &u = 0, \Delta u = 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{0.1}$$

where $u^+ = max\{u, 0\}, c$ is not an eigenvalue of $-\Delta, s \in \mathbf{R}$, and Δ^2 denotes the biharmonic operator. Throughout this paper, we assume that b is a bounded real number. Equations with nonlinearities of this type have been extensively studied in the context of second order elliptic operators (cf. [6]).

In section 1, we introduce the Banach space spanned by eigenfunctions of $\Delta^2 + c\Delta$ and investigate properties of it in the Banach space.

In section 2, we study the positive solutions of (0.1) when $\lambda_1 < c < \lambda_2, b < \lambda_1(\lambda_1 - c)$ and $s > 0$.

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1. The Banach space spanned by eigenfunctions

In this section we investigate the multiplicity of solutions of the biharmonic equation under the Dirichlet boundary condition

$$\begin{aligned} &\Delta^2 u + c\Delta u = bu^+ + s \qquad \text{in } \Omega, \\ &u = 0, \Delta u = 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where c is not an eigenvalue of $-\triangle, s \in \mathbf{R}$. Here we assume that the nonlinearity bu^+ crosses eigenvalues of $\Delta^2 + c\triangle$.

Let $\lambda_k(k = 1, 2, \cdots)$ denote the eigenvalues and $\phi_k(k = 1, 2, \cdots)$ the corresponding eigenfunctions, suitably normalized with respect to $L^2(\Omega)$ inner product, of the eigenvalue problem

$$\begin{aligned} &\Delta u + \lambda u = 0 \qquad \text{in } \Omega, \\ &u = 0 \qquad \text{on } \partial\Omega, \end{aligned}$$

where each eigenvalue λ is repeated as often as its multiplicity. We recall that $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots, \lambda_i \to +\infty$, and that $\phi_1(x) > 0$ for $x \in \Omega$.

Hence the eigenvalue problem

$$\begin{aligned} &\Delta^2 u + c\Delta u = \mu u \qquad \text{in } \Omega, \\ &u = 0, \Delta u = 0 \qquad \text{on } \partial\Omega, \end{aligned}$$

has infinitely many eigenvalues

$$\mu_k = \lambda_k(\lambda_k - c) \qquad k = 1, 2, \cdots,$$

and the corresponding eigenfunctions $\phi_k(x)$.

The set of functions $\{\phi_k\}$ is an orthonormal base for $L^2(\Omega)$. Let us denote an element u, in $L^2(\Omega)$, as

$$u = \Sigma h_k \phi_k, \qquad \Sigma h_k^2 < \infty.$$

Now we define a subspace H of $L^2(\Omega)$, which will contain all solutions of equation (1.1), as follows

$$H = \{u \in L^2(\Omega) : \Sigma|\lambda_k(\lambda_k - c)|h_k^2 < \infty\}.$$

Then this is a complete normed space with a norm

$$|||u||| = [\Sigma|\lambda_k(\lambda_k - c)|h_k^2]^{\frac{1}{2}}.$$

Since $\lambda_k \to +\infty$ and c is not an eigenvalue of $-\Delta$, we have the following simple properties of the Hilbert space H.

PROPOSITION 1.1. *Let c be not an eigenvalue of $-\Delta$. Then we have:*

(1) $\Delta^2 u + c\Delta u \in H$ *implies* $u \in H$.
(2) $|||u||| \geq C\|u\|_{L^2(\Omega)}$ *for some* $C > 0$.
(3) $\|u\|_{L^2(\Omega)} = 0$ *if and only if* $|||u||| = 0$.

Proof. (1) Suppose c is not an eigenvalue of $-\Delta$. We write

$$\Delta^2 u + c\Delta u = \Sigma \lambda_k(\lambda_k - c)h_k\phi_k.$$

Then

$$\begin{aligned}\infty > |||\Delta^2 u + c\Delta u|||^2 &= \Sigma|\lambda_k(\lambda_k - c)|(\lambda_k(\lambda_k - c))^2 h_k^2 \\ &\geq \Sigma C|\lambda_k(\lambda_k - c)|h_k^2 = C|||u|||^2,\end{aligned}$$

where $C = inf_k\{|\lambda_k(\lambda_k - c)| : k = 1, 2, \cdots\}$.
(2) and (3) are trivial. □ □

LEMMA 1.1. *Let c be not an eigenvalue of $-\Delta$. Suppose d is not an eigenvalue of $\Delta^2 + c\Delta$ and $u \in L^2(\Omega)$. Then $(\Delta^2 + c\Delta - d)^{-1}u$ belongs to H.*

Proof. Suppose that d is not an eigenvalue of $\Delta^2 + c\Delta$ and finite. We know that the number of $\{\lambda_k(\lambda_k - c) : |\lambda_k(\lambda_k - c)| < |d|\}$ is finite, where $\lambda_k(\lambda_k - c)$ is an eigenvalue of $\Delta^2 + c\Delta$. Let $u = \Sigma h_k\phi_k$. Then

$$(\Delta^2 + c\Delta - d)^{-1}u = \Sigma \frac{1}{\lambda_k(\lambda_k - c) - d} h_k\phi_k.$$

Hence we have the inequality

$$|||(\Delta^2 + c\Delta - d)^{-1}u||| = \Sigma|\lambda_k(\lambda_k - c)|\frac{1}{(\lambda_k(\lambda_k - c) - d)^2} h_k^2 \leq C\Sigma h_k^2$$

for some C, which means that

$$|||(\Delta^2 + c\Delta - d)^{-1}u||| \leq C_1\|u\|_{L^2(\Omega)}, \qquad C_1 = \sqrt{C}.\square$$

□

With Lemma 1.1, we can obtain the following lemma.

LEMMA 1.2. *Let $f \in L^2(\Omega)$. Let b be not an eigenvalue of $\Delta^2 + c\Delta$. Then all solutions in $L^2(\Omega)$ of*

$$\Delta^2 u + c\Delta u = bu^+ + f(x) \qquad \text{in } L^2(\Omega)$$

belong to H.

With aid of Lemma 1.2, it is enough to investigate the existence of solutions in the subspace H of $L^2(\Omega)$ of (1.1).

Let $\lambda_k < c < \lambda_{k+1}$ and $\lambda_k(\lambda_k - c), \lambda_{k+1}(\lambda_{k+1} - c)$ be successive eigenvalues of $\Delta^2 + c\Delta$ such that there is no eigenvalue between $\lambda_k(\lambda_k - c)$ and $\lambda_{k+1}(\lambda_{k+1} - c)$. Then $\lambda_k(\lambda_k - c) < 0 < \lambda_{k+1}(\lambda_{k+1} - c)$ and we have the uniqueness theorem.

2. Existence of positive solution

Now, we investigate the existence of the positive solution of (1.1).

LEMMA 2.1. *Let $\lambda_1 < c < \lambda_2, b < \lambda_1(\lambda_1 - c)$ and $s > 0$. Then the unique solution of the linear problem*

$$\begin{aligned} &\Delta^2 u + c\Delta u = bu + s \qquad \text{in } \Omega, \\ &u = 0, \Delta u = 0 \qquad \text{on } \partial\Omega \end{aligned} \tag{2.1}$$

is positive.

Proof. Let $\lambda_1 < c < \lambda_2$ and $b < \lambda_1(\lambda_1 - c)$. Then the problem

$$\begin{aligned} &\Delta^2 u + c\Delta u - bu = \mu u \qquad \text{in } \Omega, \\ &u = 0, \Delta u = 0 \qquad \text{on } \partial\Omega \end{aligned}$$

has eigenvalues $\lambda_k(\lambda_k - c) - b$ and they are positive. Since the inverse $(\Delta^2 + c\Delta - b)^{-1}$ of the operator $\Delta^2 + c\Delta - b$ is positive, the solution $u = (\Delta^2 + c\Delta - b)^{-1}(s)$ of (2.4) is positive.

This proves the lemma. □ □

An easy consequence of Lemma 2.1 is

THEOREM 2.1. *Let $\lambda_1 < c < \lambda_2, b < \lambda_1(\lambda_1 - c)$ and $s > 0$. Then the boundary value problem (2.1) has a positive solution u_1.*

Proof. The solution u_1 of the linear problem (2.1) is positive, hence it is also a solution of (1.1). □ □

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