Journal of the Korea Society of Mathematical Education Series D: Research in Mathematical Education
Vol. 1, No. 2, November 1997, 127–137

韓國數學敎育學會誌 시리즈 D:
<數學敎育硏究>
제 1 권 제 2 호 1997 년 11 월, 127–137

Mental Counting Strategies for Early Arithmetic Learning[1]

KOH, SANG SOOK
Department of Mathematics Education, College of Education,
Chonnam National University, Gwangju 500-757, Korea

국문 초록(Abstract in Korean): 수세기는 초등 수학교육의 기초로서 보통 유치원 과정 이전부터 시작된다. 그러나, 서수와 기수의 구별된 사용의 중요성은 미국의 "학교 수학의 교과 과정과 평가 기준" (NCTM 1989)에서 뿐만 아니라 학교 교육의 현장에서도 많이 간과되고 있는 실정이다. 일반적으로 사용되는 수직선 (Number line)과 다르게 구조적으로 개발된 Hasse's structured number line 을 사용하여 학생들에게 수세기의 의미와 기술을 가르친다면 구체적 경험을 통해 수학적 사고 능력을 키우고 개발하는데 도움이 된다. 만약 Hasse 의 9 가지 수준에 따라 다양한 학습 활동을 개발하여 수업 계획을 세워서 학습을 진행한다면 수업은 역동적이며 매우 흥미로워 질 것이다. 학생들은 말로 나타내기(Verbalization)와 상상(Imagination)의 충분한 경험을 바탕으로 정신적 표현(Mental representation)을 개발하여 수세기 기초를 확립하고 나아가 연산을 쉽게 수행할 수 있을 것이다. 여기에 소개된 교구들과 학습 활동들은 초등 수학교육이 암기 위주의 문답식이 아니며 얼마나 역동적이고 흥미로울 수 있나를 보여준다.

INTRODUCTION

The "Curriculum and Evaluation Standards for School Mathematics" (NCTM 1989) emphasizes that both the cardinal and the ordinal aspect of number are integral components of well-understood number meanings (p. 39). Nevertheless, most of the elaboration and examples in the Standards relate to cardinal and not to ordinal meanings of number. For example, in the illustration of different possible meanings for "seven" (p.39), three of the four stated meanings are cardinal meanings: "one less then 8," "one more than 6", and "5 and 2 more". Ordinal relations such as "after 6 (the 6th)" and "before 8 (the 8th)" seem to be overlooked.

NCTM (1989) emphasis on cardinality is also reflected in the choice of suggested manipulatives; regular and irregular arrays of collections of objects prevail. Linear arrangements appear in measuring contexts (p. 37) and in counting contexts (p. 39). An

[1] Part of the project directed by Dr. Heide Wiegle during the year 1995/1996 in the University of Georgia. The author actively participated at this project during her graduate school.

example that draws on position and addresses the aspect of ordinal number meaning is the game "Guess My Number" (p. 40). The students are asked to guess a number identified on a number line 0–500. This number line is unstructured except for dots indicating the position of the hundreds.

The neglect of the ordinal number aspects in favor of cardinal number meaning is common in classrooms. Order relations are only explored with respect to numerals: students have to write the numerals in sequence or they have to fill in missing numerals between the given ones. The question "Which one?" is barely asked: the focus is always on "How many?"

What could be gained by an early and equal focus on both cardinal and ordinal number meanings? First, an ordinal number word refers to a position of one object among other objects.

This focus on one object is similar (but not identical) to the act of tagging an object with a number word during counting. In a counting activity, the child creates a perceptual unit item and correlates this item to a specific number word from the number word sequence (Steffe, von Glasersfeld, Richards & Cobb 1983). A focus on position and ordinality can strengthen and capitalize on this one-to-one correlation.

Second, the question, "Which one?" allows a more dynamic and concrete approach to computing addition and subtraction facts. As a case in point, let's compare the instruction "Count back to subtract" with the more concrete and meaningful question "Which birds are flying out of the hat?" In both cases, the student is counting back to find the solution in a subtraction problem.

There is, however, a fundamental difference in how the student is counting back. For the subtraction problem 7–3, if a teacher instructs the student to say the number words "six, five, four", then the last number word, four, is only emphasized. That is, at each step, the focus is on the result: "There are six left, then there are five left, then there are four left". In contrast, the student answering the question "Which one is flying away?" starts the backwards number word sequence with seven: "the seventh, the sixth, the fifth; there are four left", or alternately, "seven, six, five, there are four left". In this case, the result of the action is only addressed in the last step, and the primary focus is on the action and on the item to which the action is applied, not on the result.

Meaning and understanding will not count for much if students do not show some proficiency in producing answers. Knowing how the multiplication algorithm works does not help students if they can not produce the multiplication facts to execute the algorithm. So we need to develop both skills and meaning. Skills and meaning are two sides of a coin; one is not possible without the other. The approach of mental counting strategies attempts to develop meaning and skills simultaneously. Avoiding relying on memorization, it employs the power of students' thinking by developing strategies based on extended concrete experiences.

In addition, number sentences such as 5 + 3 = 8 are highly abstract mathematical entities that point to a whole range of additive situations. If we want students to make sense of such an abstract sentence, they must have constructed a solid understanding of what addition is all about. In order to gain this understanding and to make the necessary abstractions and generalizations, we need to give the students enough experience of acting in additive situations. If we ask students too early to read or write abstract number sentences, we deprive students of this experience and we may foster an attitude that mathematics is a meaningless enterprise consisting only of symbols taught for performing on tests.

The Structured Number Line

The focus on actions on objects as the basis for students' construction of the arithmetic operations is one of the principles of an arithmetic course developed by the German educator Hermann Hasse at the beginning of the 20th century (1911, referenced in Breidenbach 1963), and widely used in German elementary schools up to the time of the new mathematics.

Haase built early experiences with counting, addition, and subtraction around a manipulative that combines cardinal and ordinal aspects of number based on 9 levels in his basic course. The manipulative consists of a structured number line and objects that can be attached to the number line. The basic unit of the number line is 10 pegs mounted on a piece of wood. Special features of the structured number line are the absence of a representation for zero and the structure given by the different heights of the 5th and 10th pegs (Figure 1).

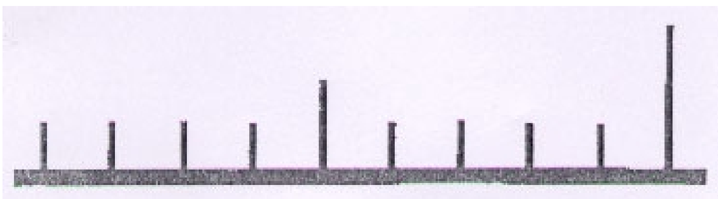


Figure 1. The basic unit of the structured number line

The structured number line and number-intensive activities that can be related to the number line give students opportunities to develop strategies that supplement and go beyond finger counting and rote memorization. Students are encouraged to solve arithmetic number sentences by acting and concurrently describing what they are doing. With repeated experience, students are able to describe possible actions without having to perform them: they solve arithmetic number sentences by reasoning. In Piaget's terminology, it can be said that Haase's method builds on students' internalized actions.

BUILDING THE NUMBER LINES

A class model of the basic unit of the structured number line can be built using two 40 cm long, 4 by 4 cm pieces of wood. The two pieces are attached by a hinge. The pegs are cut from dowels (1 cm diameter) available in any hardware store. The pegs corresponding to the numbers 1 to 4 and 6 to 9 are 10 cm high, the 5th peg is 15 cm high, and the 10th peg is 18 cm high.

In addition to the class model, each child or pair of children should have a working model of the number line. A drawn representation of the number line is copied to a letter-size sheet of cardstock which can be laminated. The individual models can either be taped to each table or stored in a central location. An overhead transparency of the number line is also useful. The structured number line itself provides a backdrop for the actual manipulatives such as blocks, unifix cubes, stickers, or other counters. Ideally, the blocks for the class model are rather large and have holes, so that the students can slide them over the pegs (see Figure 2).

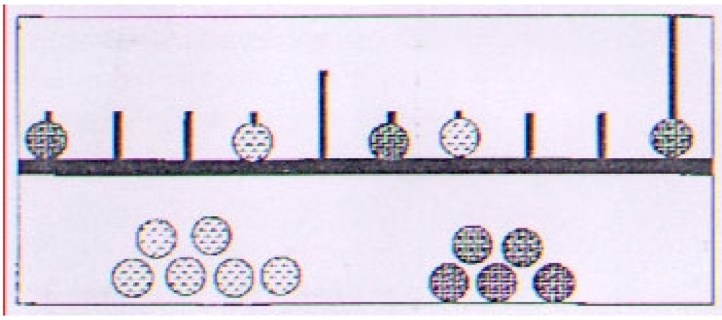


Figure 2. Basic unit of the number line with manipulatives

For the individual models, students can use stickers or small counters. Counters have the advantage over stickers that they are more concrete and can relate directly to the situations being modeled.

Stickers have the advantage over counters that the results of the students' work can be collected for evaluation because the stickers are attached to the number line. The stickers can be removed easily if the number line is laminated.

The number line can be extended to 20 (grade k-1), 100 (grade 1–2), and 1000 (grade 3 +). Two basic units of the number line taken side by side represent the numbers from 10 to 20. The units of the number line up to 100 are usually smaller versions of the basic unit (see Figure 3), and the blocks are smaller as well. The pegs for 50 and for 100 are taller than the pegs corresponding to the other 10's; thus, the structure of the basic unit is recursively applied to the number line representing the numbers 1–100. If the construction of a wooden model of the number line 1–100 is not feasible, the teacher can assemble 10 individual cardstock units 1–10 on a wooden panel and then mount the panel to a board.

Figure 4 shows part of the number line in front and side view. The blocks are placed in front of the number line (see side view). Alternately, the 10 basic units can be mounted on poster board and then attached to a wall. In this case, the students can only work with stickers. Sometimes, a poster board model is the only feasible option for a number line 1–1000. Usually, this number line stretches over more than one wall.

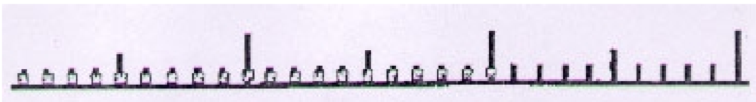
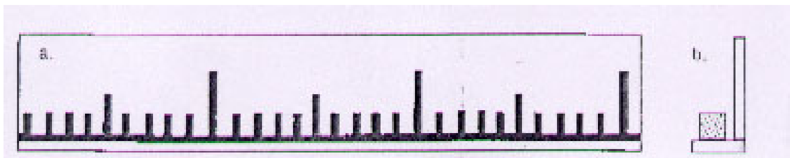


Figure 3. Part of the structured number line 1–100



a. frontview, b. sideview

Figure 4. The drawn number line mounted to a panel.

Counting Activities

The following activities are suggestions of how to introduce the number line in grades 1 or 2, after the students have already experienced a considerable amount of mathematics instruction. The activities can be used with the whole class or with individual students. The initial counting activities should then be extended to additive and take-away situations (see levels 4–9, pp. 10–12).

The class model: In using the number line, students are intrigued by the wooden class model of the structured number line. They comment on the different heights of the pegs, and they spontaneously count the pegs. Some students explore how the pegs sound as a musical instrument. Eventually, the teacher introduces the convention that the highest peg is “the tenth”. At this point, the teacher gets one of the blocks, slides it over the first peg and thus mentions the “first”.

Initial activities serve to make the students familiar with the number line 1–10 and address the students’ knowledge of and flexibility with the number word sequence. For example, students are asked to identify certain pegs:

“Put a ball (block) on the first peg.”
“Put a ball (block) on the fourth peg.”
“Put a ball (block) on the last peg. Which one is that?”

Later, the relationships “after”, “before”, and “between” are addressed:

“Put a ball (block) right after the fifth.”
“Which one is right before the tenth?”
“Put a ball (block) between the sixth and the eighth.”

The structure provided by the different heights of the fifth and the tenth pegs helps students to access the number word sequence at various points other than “one”. For example, Tom, a kindergarten student who had to start at “one” when asked to “count from eight to eleven” (without manipulative), would identify the eigth peg by counting “five, six, seven, eight”. The ability to access the number word sequence at various points without having to fall back to “one” every time is an important prerequisite for a later curtailment of the counting activity by counting on.

The individual models: Parallel to the placement of a block or ball on the class number line, students can place smaller objects on the pegs of their individual number lines. For individual or part work, the teacher can give the students a sheet with stickers, which contain a numeral. The students then place blocks on the pegs as indicated by the numerals on the stickers. Alternately, they can use the stickers directly. If the teacher wants to assign homework, she can prepare a worksheet containing a drawn number line and several numerals. The students then draw the blocks, balls, or other objects as indicated by the numerals (see Figure 5).

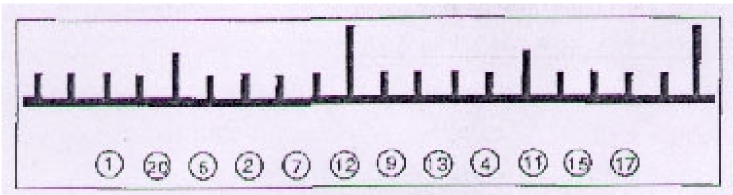


Figure 5. Individual model for seat work or homework

Usually, students are not asked to assign objects to all 10 or 20 pegs, nor to write all the numerals in order. The goal for the students is flexibility and a good intuition about where to place an object on the number line. We do not want them to fall into routines that require little thinking. Instead, we want the students to develop a mental picture of the number line.

To this end, students can be asked to close their eyes and the teacher narrates, for example, “walk all the way from the first to the seventh peg” or “look closely at the fifth peg and place a block on the peg right before (after) the fifth”. When the students open their eyes they can demonstrate the mental action that they imagined on the class or individual number lines.

The overhead model: A number line on an overhead transparency also provides opportunity to strengthen the students’ imagination and mental representation. Before the projector is turned on, the teacher can place one of the counters on the number line. The projector is turned on for, at most, two seconds, to show the configuration to the class and immediately turned off.

This can be repeated if the students need a second look. The students then can describe how many counters they saw and how they saw them, or they can demonstrate the configuration on their individual number lines. The overhead activity is similar to an activity with 10-frames suggested by Cobb and Merkel (1989).

Combining Cardinal and Ordinal Aspects

If the teacher asks students to place two (three, four) blocks on the number line, the students have various possibilities to respond. Some may automatically place the two (three, four) blocks on the first few pegs. But other students may choose different pegs. As the students present their solutions, they have ample opportunity for verbalization and imagination.

For example, each student’s solution can be built on the class number line, the overhead number line, or on the individual number lines. Eventually, the teacher will introduce the restriction that the two blocks have to “be together”, that is, on adjacent pegs.

A student may explain his or her configuration as “I have two blocks, they are on the seventh and eighth pegs”, or “My blocks are on six, seven, and eight”. The activity of placing a small number of blocks on different pegs and describing the configuration provides a foundation for verbal or mental counting activities that use auditory patterns of two, three, and four to keep track of the individual counting acts.

CARDINAL AND ORDINAL NUMBER WORDS

If an adult counts a collection of objects, each number word (e.g., “five”) attached to an item has an ordinal and a cardinal meaning. In the ordinal meaning, the number word identifies the position of the item counted; in the cardinal meaning, the number word denotes how many items have been counted so far. The adult can determine the cardinal or ordinal meaning of the number word by the context, and he or she can switch between both meanings without problem. Ordinal number words beyond “first”, “second”, and “third” are usually not used. The teacher has to decide whether to introduce ordinal and cardinal number words beyond the first three to help the students distinguish between ordinal and cardinal meanings.

WORKING WITH NUMBER LINES UP TO 100 AND 1000

Activities promoting flexibility with the number word sequence and addressing the positional aspects of numbers are appropriate throughout the elementary school years; they should be part of mathematics lessons on a regular basis. Sample tasks are:

Show 50 pegs.
Show the 50th (50 번째) peg.
Show five tens (십이 다섯 개).
Show the fifth ten (5번째 십).

Show 300 pegs.
Show the 300th (300 번째) peg.
Show three hundreds (백이 세 개).
Show the third hundred (3번째 백).
Show 30 tens (십이 30 개).
Show the 30th ten (30번째 십).

Number line activities should not replace but rather complement bundling activities, work with 10-frams, stickers of unifix cubes, the 100-chart, bean stickers or base 10 blocks, and place-value mats.

OVERVIEW OF LEVELS OF HAASE’S BASIC COURSE 1–10

Level 1: Numbers 1 to 4: Identifying what is counted

* two: one and another one
* three: one, another one, and another one
* four (two twos): one, another one (pause), another one and another one
* one

(Notice: “One” is introduced last and “Zero” is not introduced.)

Level 2: Addition and Subtraction: Concrete work with numbers 1 to 4, 5, or 0

* Continued exercises from Level 1.
* Additive and take-away situations with action words such as “flying to the bird house”, “joining children in the playground”, “running away”, or “eating”
* Perceptual recognition of the sum and difference as well as identifying what is added or taken away (“Two birds are in the house; one more bird arrives, now we have three birds in the house”). It is important that the students actually do the problems, either by playing “birds” or by using some representation such as felt figures, unifix cubes, or blocks.

(Notice: Five and Zero can be introduced.)

Level 3: Introduction of the structured number line; ordinal number words 1 to 4

* Getting used to placing objects from left to right
* Ordinal number words: first, second, third, fourth
* Ordinal versus cardinal number meaning

Level 4: Addition and Subtraction (numbers 1–4)

* Which one? Acting, speaking, and thinking
* Additive situations: “Two houses are on the table; get two more houses, and tell the class which ones you are bringing”. “Two houses are on the table, I get two more, the third and the fourth, now there are four houses all together”.
* Take-away situation: “Four cars are in the parking lot, two cars are driven away. Tell the class which cars are driven away”. “Four cars are in the parking lot, the fourth is driven away and then the third. Two cars are left.”

Level 5: Numbers 1–10

* Extension of number concepts to 10; ordinal and cardinal meanings and number words

Level 6: Working with numbers 1–10, addends and subtrahends 1–4

* Which one? Acting, speaking, and thinking.
* Extension of strategies from level 4 to numbers 1–10.

Level 7: Numerals and operation signs; number sentences

* Number sentences such as 5+3=8 are a short way of writing about additive and

take-away situations

Level 8: Working with numbers 1–10; addends and subtrahends 5–9

* Commutative property for addition (switching the numbers around) verbalized
* Small numbers (1–4) and big numbers (5–10)
* Small numbers go to the back, large to the front (addition)
* Small numbers are taken away from the back, large from the front (see Figure 6)

Level 9: Extensions

* More than one addend / subtrahend;
* Missing-addend / missing-subtrahend problems

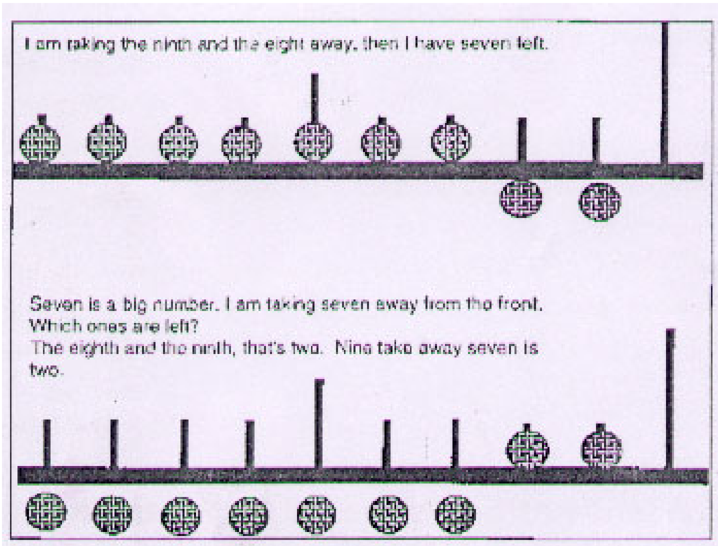


Figure 6. Subtraction of small (1–4) and big (5–10) numbers

REFERENCES

Breidenbach, W. (1963): Rechnen in der volksschule (Sechste, neu bearbeitete und erweiterte Auflage) [*Arithmetic in the general school* (6th, revised and extended edition)]. Hannover, Germany: Hermann Schroedel.

Cobb, P. & Merkel, G. (1989): Thinking strategies: Teaching arithmetic through problem solving. In: P. R. Trafton (Ed.), *New directions for elementary school mathematics.* (1989 Yearbook), pp. 70–81. Reston, VA: National Council of Teachers of Mathematics.

NCTM (National Council of Teachers of Mathematics) (1989): *Curriculum and evaluation standards for school mathematics*. Reston, VA: NCTM.

Piaget, J. (1964): *The child's conception of number,* 2nd ed. (C. Gattegno & F. M. Hodgson, Trans.). London: Routledge & Kegan Paul. [Original work published in 1941]

Steffe, L. P; von Glasersfeld, E.; Richards, J. & Cobb, P. (1983): *Children's counting types: Philosophy, theory, and application*. New York: Praeger.