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Fuzzy Concept and Mathematics Education

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G. Cantor: Das Wesen der Mathematik liegt in ihrer Freiheit. (Freedom is the essence of mathematics.)

One of the main objectives of school mathematics education is to develop a student' intuition and logical thinking [11]. But two-valued logical thinking, in fact, is not sufficient to express the concepts of a student's mind since intuition is fuzzy. Hence fuzzy-valued logical thinking may be a more natural way to develop a student's mathematical thinking.

BACKGROUND

Byung-soo's Story: A few years ago, I used to meet with an 89 year old man while hiking in Kumjung mountain near my house. He always walked very slowly and carefully so that he would not fall.

One day, he said to me that he envied my energetic way of walking.
Then he asked me, "How old are you?
I replied, "I am 44."
He said, "Then you are a junior-old-man!"
"What did you say?, Old man?"

His talk gave me a big shock, because until then, I had never thought of myself as an old man.

My long standing observation from this conversation is that there is difference in how people view the world. For example, there can be doubt whether we call someone old and young. And that this vagueness, rather than something to strive to do away with, is a permanent part of life.

I. INTRODUCTION

Existing mathematics, in fact, can be called two-valued mathematics since it is based on two-valued logic which is used as a language to represent what is described by mathematics. On the other hand, our life including our mind and language, does not follow two-valued logic but multi-valued logic. So the existing two-valued mathematics is not appropriate and sufficient to describe our life and express our mind satisfactorily and effectively.

Moreover, many well-established fields such as psychology, pedagogy, sociology, epistemology, cognitive science, semiotics and economics, which are bases and instruments for teaching and learning mathematics, are all expressed in a natural language rather than the formal language of mathematics.

Therefore, we should find a modified and generalized mathematics substituted for two-valued mathematics, which can express our mind and life more completely or precisely. In fact, because logic plays a central role in mathematical systems, as a language a multi-valued logic in place of two-valued logic would help to construct a more adjustable mathematics.

Also it would help our students adapt themselves to the changeable world. To do all of this, existing mathematics should state its theory in terms of multi-valued logic, and an existing mathematics education should not neglect the development of a student's intuition.

In this paper, first we consider the weakness of, so called, two-valued mathematics and two-valued mathematics education, and then we discuss the meaning of natural language. Next, we introduce the concepts of a fuzzy set which is a fuzzy version of an ordinary set, and fuzzy logic which is an infinite-valued logic incorporated using fuzzy sets, which also generalizes the two-valued logic. Finally, we give some background educational philosophy on the problem of why mathematics education should be improved. As one prescription to this problem, we are going to suggest the adoption of fuzzy logic in the mathematics curriculum.

After this, our main concerns are as follows:

(1) What sorts of didactics would be relevant for teaching fuzzy mathematics and fuzzy mathematical knowledge?
(2) What problems would arise if we taught fuzzy mathematics instead of two-valued mathematics?
(3) What sorts of fuzzified methods would be relevant for teaching two-valued mathematics and two-valued mathematical knowledge?

II. Weakness of mathematics

There is no unique answer to the question "what is real mathematics?"

Usually descriptions of the nature of mathematics are dependent on the viewpoints adopted. Mathematics can be seen variously as a body of knowledge, a collection of techniques and methods, the product of human activity and even as the activity itself; namely, the solving of problems [7]. One viewpoint is that of Hirst and Peters [10] who regarded mathematics as one of the seven forms of knowledge. They dealt with mathematics as an impersonal body of knowledge.

Mathematics is mainly characterized by its distinctive concepts, propositions and the methods of verifying its propositions, namely logical proof. And it can be said that a mathematical system is composed of an undefined concept, an ordinary set, algebra of sets, relation, operation, rule of reasoning, logical axiom, non-logical axiom, definition and theorem.

But mathematics is more than a collection of theorems, definitions, problems and techniques: it is a way of thought [18]. Mathematics as an expression of the human mind reflects the active will, the contemplative reason, and the desire for aesthetic perfection [6]. Many considerations which may influence the way mathematicians think and act such as intuition, agreement with empirical evidence, taste, esthetics, wishful thinking, personal ambition are not two-valued. Wittgenstein [22] says that what we call mathematics is a family of activities with a family of purposes. He sees mathematics as consisting of a motley of human activities driven by a range of human goals, intentions and purposes. In addition, there is always a central core of mathematical reasoning that is supposed to be logically sound. So, it is fairly reasonable and intuitive to consider multi-valued logic or fuzzy logic. If our conversation were to be two-valued rather than fuzzy valued, then our daily life would be more difficult.

For example, if we have a green piece of cloth with a bluish tint we may be in doubt whether we should call the color green or blue-green, and we may even disagree about the name we wish to give to the color. Similarly we can not say that a person is either tall or not tall. Even if we give an artificial definition to "tallness" we may run into trouble because no measurement is absolutely precise.

Even though ordinary mathematics deals with concepts subject to the rules of two-valued logic, in particular to the postulate of the excluded middle, it is not true that all statements involve concepts that are subject to logic.

Magnus [12] said that as far as he knows, Nietsche was the first to point out the fuzzy concept. Nietsche claimed that only man-made concepts are subject to logic. But classical mathematical logic divided the world into "yes and no", "white and black", "true and false". Actually, since not all sentences involve concepts that are subject to two-valued logic, we have to deal with many kinds of sentences that come from our lives and thoughts, and ordinary mathematical logic cannot handle them thoroughly.

Concepts of the classical logic need to be changed to reflect and express reality more effectively. It must be realized in mathematics that reality is more or less uncertain, vague, and ambiguous.

III. WEAKNESS OF MATHEMATICS EDUCATION

Mathematics based on two-valued logic and formalized language is said to be a proper subset of the social sciences (sociology, pedagogy, psychology, economics, etc.) which are based on multi-valued logic and natural language as shown in the following diagram.

Mathematics $\longleftrightarrow$ (0,1)

social sciences* $\longleftrightarrow$ [0,1]

$\longleftrightarrow$: isomorphic correspondence

* sociology, education, psychology, economics, pedagogy, etc.

And mathematics education lies at the crossroads of many well-established scientific fields such as mathematics, sociology, pedagogy, psychology, epistemology, cognitive science, economics, and semiotics. Therefore it is concerned with problems imported from these fields.

Hence mathematics education has a point of weakness, since it deals with infinite-valued subjects in a two-valued way. It is the same as when we watch an ocean from one place on a beach.

The widespread public image of mathematics is that it is difficult, cold, inhuman, abstract, theoretical, ultra-rational, and relates absolutist philosophies of mathematics. It is argued that this image is consistent with separated values. In contrast, an opposing humanized image of mathematics, consistent with connected values, finds academic support in recent *fallibilist* philosophies of mathematics. It is argued also that although these two philosophical positions have a major impact on the ethos of the mathematics classroom, there is no direct logical connection. It is concluded that the values realized in the classroom are probably the dominant factors in determining the learner's image and appreciation of mathematics [8]. Ernest's conclusion mentioned above is based on an existing mathematics which, is bivalent. However, a humanized image of mathematics, which is consistent with connected values, is not bivalent.

For a long period of time, research in mathematics education has been profoundly influenced by Piaget's genetic epistemology and development psychology. The object of study in mathematics education might be, for example, the teaching of mathematics; the learning of mathematics; teaching and learning situations; the relations between teaching, learning and mathematical knowledge; the reality of mathematics classes; societal views of mathematics and its teaching; or the system of education itself [16]. The teaching-learning process is considered as a social interaction.

Because the teaching-learning process implies an interpersonal relationship, and interpersonal relationships are concep-tualized and studied typically in sociology, mathematical meaning is taken as a product of social process, in particular as a product of social interactions [20]. From this point of view, mathematical meanings are primarily studied as emerging between individuals, not as constructed inside or as existing independently of individuals.

Learning is a process of knowledge construction, not of knowledge recording or absorption [1], and the learner is aware of the processes of cognition and can control and regulate it; this self-awareness or metacognition significantly influences the course of learning [9]. Learning is not a passive receiving of ready-made knowledge but a process of construction in which the students themselves have to be the primary actors [21]. Thus mathematics education should help the learners to be active and to use their intuition. intuition and logical thinking are complementary and closely interrelated in human mathematical thinking. In other words, human thinking could develop productively and soundly only when intuition and logical thinking are in a harmonious and cooperative relation [11]. In addition, because the concept of intuition is fuzzy, it is more appropriate to use fuzzy logic with the concept of intuition than two-valued logic. To do that, mathematics education must be conducted for students, and teachers must listen to their difficulties in learning mathematics. Thus to help the students, the most important thing for teachers is to let their students use natural language in learning mathematics in a constructivist learning environment, and learn fuzzy mathematics which is consistent with social sciences.

IV. Meaning of natural language

The traditional view of formal language is that ordinary language contains ambiguous notions, which is the main cause of many philosophical problems. Such problems take the shape of linguistic misunderstanding. To prevent such confusions, it is useful to introduce a formal language.

But, Wittgenstein refused to accept the following assumption; formal language is able to give the best picture of reality possible. Rather he thought it was possible that natural language gave the best picture of reality.

Contrary to this assumption, natural language philosophy has emphasized that a formal language only gives a very rough and imprecise picture of reality. The richness of natural language is essential because it creates a descriptive potential which can never be achieved by formal description; and such nuances are necessary for serious communication.

A formal language might a dead and very imprecise plaster cast of a part of natural language. Natural language is based on common sense interpretations of reality. In addition, the nucleus of the student's constructive activity consists of building meanings associated with his or her own experience, including linguistic experience.

The socialization of this process consists of negotiating these meanings in a community —the classroom —which has assumed this constructive process for itself. The sensation of objectivity arising from the process of negotiation can induce the belief that this shared knowledge pre-exists the community constructing it. For this reason, the relationships between mathematics and language must be analyzed with care. In fact natural language is a field of experimentation for the student [14]. The discussion of the formatting power of mathematics cannot progress unless we have a natural language description of a situation in which the formatting take places.

Hence for the development of the formatting power of mathematics, the act of formalizing a natural language is an important epistemological step. Since our thought and life is based on natural language, we need to handle natural language within our mathematical concept category.

Hence the fuzzy concept and the natural language would be incorporated in our mind, extend fuzzy mathematics as a generalization of two-valued mathematics and generalize mathematical education to deal with mathematics with fuzziness.

V. FUZZY SET AND FUZZY LOGIC

Logic plays a central role in mathematics as a language, and there is a correspondence between the logical connectives "and, or, not, implication" and the set of operations "intersection, union, complement, inclusion", respectively. It is established that this correspondence (called isomorphism) guarantees that every theorem or result in set theory has a counterpart in two-valued logic and vice versa [2]. The important primary property of an ordinary set is that either an element belongs to the set or not. Ordinary logic called two-valued logic is isomorphically connected with the ordinary set. The two concepts, ordinary sets and ordinary logic, play a central role in the mathematical system.

Fuzzy set and fuzzy logic, founded in the mid sixties by Zadeh [23], can be viewed as a broad conceptual framework containing classical set theory and classical logic. Moreover in the last three decades significant progress has been made in the development of fuzzy set and fuzzy logic theory, and their use in large variety of applied topics.

Fuzzy set and fuzzy logic are powerful mathematical tools for modeling; uncertain systems in industry, nature, the humanities; and as a facilitator for common sense reasoning in decision making in the absence of complete and precise information. Their role is significant when applied to complex phenomena not easily described by traditional mathematical methods, when the goal is to find a good approximate solution. Fuzzy logic is attracting a great deal of attention in the business and industrial world as well as among the general public [4].

In contrast to the stochastic uncertainty-type vagueness, the vagueness concerning the description of the semantic meaning of events, phenomena or statements is called fuzziness [24]. In fact, vagueness is no more to be done away with in the world of logic than friction in mechanics.

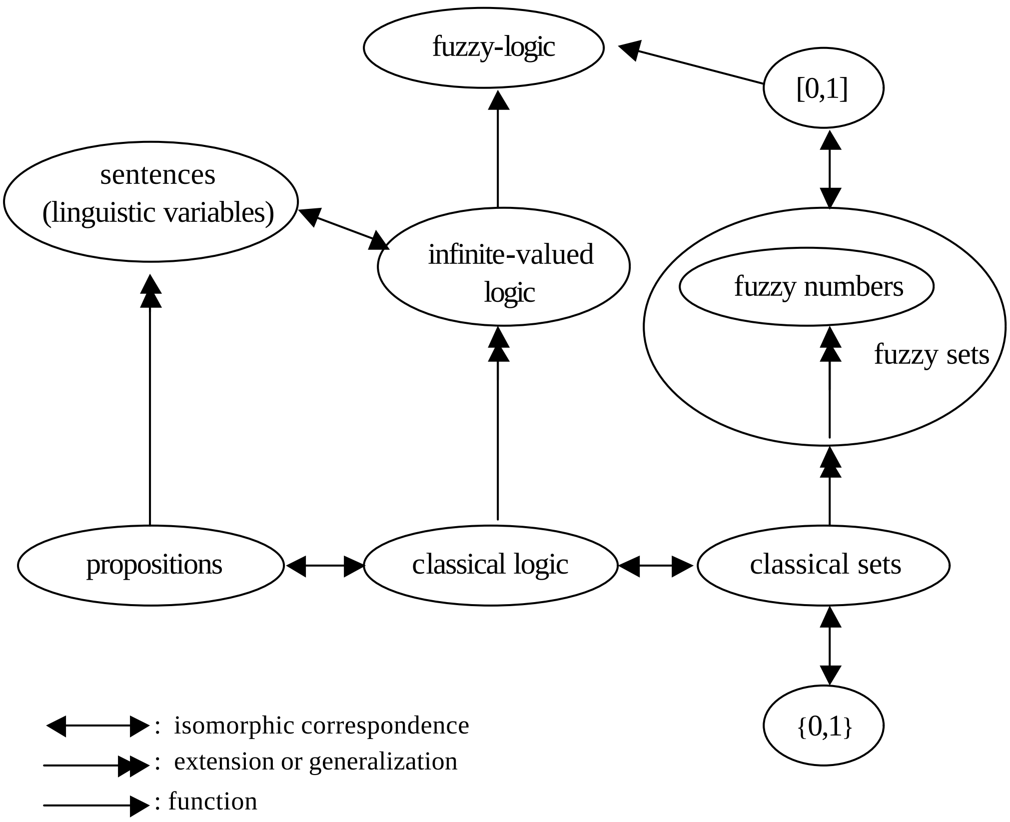
Fuzzy set and fuzzy logic have been applied to virtually all branches of science, engineering and socio-economic sciences. All traditional logic habitually assumes that precise symbols are being employed. It is therefore not applicable to this terrestrial life, but only to an imagined celestial one. The law of the excluded middle is true when precise symbols are employed but it is not true when symbols are vague, as, in fact, all symbols are.

All languages are vague. Vagueness, clearly, is a matter of degree [15]. An important step towards dealing with vagueness was made by the philosopher Black [2] who introduced the concept of vague set. That the concept of fuzziness contains that of vagueness was shown in [23].

Fuzzy logic can be considered as an extension of infinite-valued logic in the sense of incorporating fuzzy sets and fuzzy relations in the system of infinite-valued logic. Fuzzy logic uses as a major tool fuzzy set theory. It focuses on linguistic variables in natural language and aims to provide foundations for approximate reasoning and imprecise propositions.

The relations between classical sets (ordinary sets or crisp sets), classical logic (two-valued logic), fuzzy sets (in particular, fuzzy numbers), infinite-valued logic, fuzzy logic, sentences and statements are schematically shown as follows:

FUZZY RELATIONS



VI. Fuzzy concept and mathematics education

Ideas, pictures, images and value systems are formed in all human king and feeling. Human thinking and feeling have certainly more concepts or meanings than our daily language has words. It becomes quite obvious that the power of our thinking and feeling is much higher than the power of a living language. If in turn we compare the power of language with logical language, then we will find that logic looks even poorer. Therefore it seems to be impossible to guarantee a one-to-one mapping of problems and systems in our imagination to a model using a mathematical or logical language. This is because, in fact, real situations are very often not crisp and deterministic and they can not be described precisely. And the complete description of a real system often would require far more detailed data than a human being could ever recognize, process and understand.

A fuzzy set defines the degree to which an element belongs to the set. Fuzzy sets, similar to fuzzy numbers, are used to describe linguistic variables. Fuzzy sets play an important role in both applications and theoretical studies involving various types of fuzziness. They are instrumental for the development of fuzzy logic. The concept of fuzziness would be sufficient to solve those problems easily. In particular, it would be helpful in the complexity of the mathematical education system. When Zadeh [23] proposed the concept of fuzziness in 1965, in the background one can see a concealed wish to improve the relationship between humanity and the computer. Though there is a great deal in common shared between computers, which are two-valued logical machines, and the infinite-valued thinking of people, with their emotion and intuition, there are also wide differences. Fuzzy sets and fuzzy logic are communication mediums that speak to both the logical nature of the sciences and the complexity of the humanities and social sciences.

The concept of fuzziness can be found in many areas of daily life, such as in engineering [3], in medicine [19], in meteorology [5], in manufacturing [13]; and others. It is particularly frequent, however, in all areas in which human judgement, evaluation and decisions are important. These are the areas of decision making, reasoning, learning and so on.

By imprecision we mean the sense of vagueness rather than the lack of knowledge about the value of a parameter. Fuzzy set theory provides a strict mathematical framework in which vague conceptual phenomena can be precisely and rigorously studied.

The nature of mankind is essentially fuzzy and the purpose of education is to develop the nature of the mankind. But, two-valued logic and ordinary sets are artificial, rigid things which rather might prevent the development of a student's mind and intuition. So the acceptance of fuzzy mathematics and fuzzy educational methods can be said to be necessary in the classroom, especially in the mathematics education classroom.

VII. Concluding remarks

Fuzzy sets and fuzzy logic are generalizations and extensions of ordinary sets and two-valued logic, respectively. Fuzzy sets and fuzzy logic are communication media that speaks to both the logical nature of the sciences and the complexity of the humanities and social sciences. Moreover they are new logical tools those can express ambiguities, hence they are more natural and more precise.

A mathematical measurement of a wide variety of ambiguous phenomena, including the concept of probability should now be established. As a mathematical system, fuzzy sets and fuzzy logic expand the current framework and build a world that takes in new concepts, so they have interested researchers on the theoretical side from early on.

Hence it is more desirable, reasonable and advisable to teach fuzzy mathematics and fuzzy mathematical knowledge generalizing two-valued mathematics and two-valued mathematical knowledge, respectively. If it is not accepted or too difficult to teach fuzzy mathematics and fuzzy mathematical knowledge, it is more anticipative to teach ordinary mathematics and ordinary mathematical knowledge by using fuzzified educational methods.

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