Journal of the Korea Society of Mathematical Education Series D:
Research in Mathematical Education
Vol. 1, No. 1, July 1997, 13–22

韓國數學教育學會誌 시리즈 D:
<數學教育硏究>
제 1 권 제 1 호 1997 년 7 월, 13–22

Accomplishments and Prospects in the Psychology of Mathematics Learning[1]

KIRSHNER, DAVID
Department of Curriculum & Instruction, Louisiana State University, Baton Rouge, Louisiana 70803-4728, U.S.A.

Cognitive psychology has provided valuable theoretical perspectives on learning mathematics. Based on the metaphor of the mind as an information processing device, educators and psychologists have developed detailed models of competence in a variety of areas of mathematical skill and understanding. Unquestionably, these models are an asset in thinking about the curriculum we want our students to follow. But any psychological paradigm has aspects of learning and knowledge that it accounts for well, and others that it accounts for less well. For instance, the paradigm of cognitive science gives us valuable models of the knowledge we want our students to acquire; but in picturing the mind as a computational device it reduces us to conceiving of learning in individualist terms. It is less useful in helping us develop effective learning communities in our classrooms. In this paper I review some of the significant accomplishments of cognitive psychology for mathematics education, and some of the directions that situated cognition theorists are taking in trying to understand knowing and learning in terms that blend individual and social perspectives.

Thank you for the privilege and the opportunity to address the community of mathematics education in Korea. In North America the accomplishments of Korean and other Asian school children in mathematics is well known (e.g., Beaton & Mullis, et al. 1996). The ordinary achievement of Korean students is like a dream that we can only hope to achieve in some distant future in America. In part the success of Korean students is due to the importance in which schools and learning are held in Korean society. For this I salute you with admiration and with envy.

In part the success of Korean mathematics education is due to the excellence of your teachers and your professors, and for the methods that you employ in your classrooms. I have had the pleasure to meet some of the teachers and professors of Korea, and I have a tremendous appreciation of your talents and your dedication. Indeed, while I am here, I

[1] Paper presented to the Plenary Session III of the 21st National Meeting of Mathematical Education, Seongnam, Korea, July 21–24, 1997, organized by the Korea Society of Mathematical Education.

will be collaborating on research to understand more about your successful teaching methods. So it is with great humility that I address you today about some of the evolving North American perspectives on mathematics learning.

Throughout this past century, education in the United States has been especially susceptible to the influences of psychological theories. In some respects this is a good quality, because it opens American education to new ideas and methods. But the directions pointed to by psychology have not always served education well. For instance for many decades during this century behavioral psychology focussed our teachers and administrators on measurable performances, without sufficient attention to students' comprehension. This is a legacy of psychology's influence with which American educators struggle to this day. To a considerable extent cognitive psychology has helped educational theorists to refocus on the mental models and maps that students create through their learning. This is some help, but on balance we might wish for a stronger educational culture in the United States involving more internal debate about educational values. Then we could more critically evaluate the possible contributions of psycholo-gycal theories, and more deliberately select our own directions. There is an old story that we tell about education in the United States concerning a man who is down on his hands and knees looking for his lost car key underneath a lamp post. Pretty soon a stranger comes by and offers to help him look. After some time of searching with no success, the stranger asks "where did you lose your key, anyway?" The man points some distance away into the darkness and says "over there". "well, then, why are we looking for them over here?" asks the stranger. "Because the light is much better under the lamp post," replies the man. In the same way, psychological theories may shed much light on educational processes. But we cannot afford to ignore important parts of the problem just because psychology has not yet extended to those parts.

It is in this spirit of critical evaluation of psychology that I offer perspectives on the considerable benefits, as well as some of the blind spots, that cognitive psychology brings to education. It is especially heartening that the traditional relationship of influence from psychology to education is being reversed in the United States. A number of influential cognitive scientists have abandoned the methods they were trained in to pursue new approaches that can meet educational needs more effectively (e.g., Collins, Brown & Newman 1989; Greeno 1993). In many respects, this work is still in its infancy. So it is too early to begin printing new teachers manuals and textbooks. But this work can help us to think about the strengths of the theoretical tools that we have available to help guide mathematics education, and some of the areas of weakness that still need to be addressed. I count myself among those who have shifted away from a cognitive psychology orientation, and my research in Korea reflects these new interests. If time permits, I will say a little about the research in Korea.

Rather than lecturing to you about psychological theories, I would like to focus my talk around a problem that we can use to think with about learning mathematics. The problem that I offer is not a typical mathematics problem, but it is a typical logic problem that one might see on an intelligence test. I am using this problem because I like it, and because it has many elements in common with mathematics problems. I ask for your participation in thinking about this problem and sharing your thinking with me and with your fellow educators.

On the overhead there are four figures. The task is to select the figure that does not belong with the others. Take a moment to look at the figures and decide which one does not belong with the others. Now turn to your neighbor and inquire what is his or her thinking on this problem.

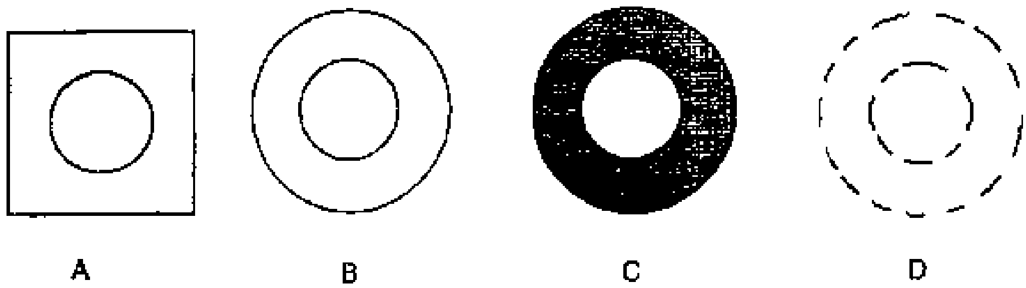

Figure 1. Which One Does Not Belong?

Is there anybody who has selected A as the answer?

Can you please explain to us why you have chosen A? A is a good choice because only in A is there a square shape. All of the other shapes are circular.

Let us take a few minutes to analyze the processes that underlie this solution. Note that this solution involves inductive (bottom up) and deductive (top down) reasoning. First, there is a need to identify and to classify features of the four figures. A typical analysis from cognitive psychology might look something like this (see Table 1):

Table 1. Features Used in Solving a Logic Problem

Figure A	Figure B	Figure C	Figure D
two shapes square shap round shap	two shapes round shape round shape	two shapes round shape round shape	two shapes round shape round shape

Once this set of categories has been established through inductive reasoning, we can deduce that A is the figure that doesn't belong. It has a unique property not shared with any of the others.

I want to suggest that a great deal of the thinking that students do in mathematics classes is of this sort. In deciding which procedure to use to solve a problem, the student must use inductive reasoning to identify the properties of that problem and to compare these with the properties of the various solution procedures that they know. Of course the school task is much harder, because the solution procedures must be retrieved from long term memory, and the properties must be coordinated without the visual support that we have available in the diagram. Cognitive psychology has made a wonderful contribution to mathematics education by providing detailed models of how such thinking is accomplished, from the inductive stage (e.g., Holland, Holyoak, Nisbett & Thagard 1989), to the deductive stage (e.g., Evans 1982), to the applications to mathematics word problems (e.g., Hinsley, Hayes & Simon 1977; Paige, & Simon 1966), to the effects of practice and automaticity (e.g., Anderson 1983).

Has anybody got a different answer to our problem? Did anybody pick D as the figure that doesn't fit with others? Could somebody who picked D share with us his or her reasoning? D is a good choice because all of the other shapes are constructed of solid lines. D is the only figure constructed of broken lines.

In this case, we can provide a similar cognitive analysis (Table 2).

Table 2. Other Features Used in Solving a Logic Problem

Figure A	Figure B	Figure C	Figure D
two shapes solid lines	two shapes solid lines	two shapes solid lines	two shapes broken lines

Well, why stop when we have such a good thing going. Does anybody have any other answers? Did anyone pick C as the figure that is unlike the others? Yes, we can see that C has the unique characteristic of being shaded in. All of the other figures are unshaded. So we have good reason to pick C.

Does anybody have any other answers to offer for this problem? Did anybody pick B as the figure that doesn't belong with the others, that is different? Can you explain your reasoning? Yes, B is a good choice, because it is the only figure that does not have a distinguishing feature to separate it from the others.

As with the others, we can give a cognitive analysis of this solution process in terms of deductive and inductive reasoning (Table 3).

Table 3. Still Other Features Used in Solving a Logic Problem

Figure A	Figure B	Figure C	Figure D
has distinguishing feature	has no distinguishing feature	has distinguishing feature	has distinguishing feature

All of the answers that have been given are good, well-reasoned solutions to our logic problem. But this last solution, B, has some unique characteristics that I would like to comment on. The first thing to notice is the relational nature of the feature "has a distinguishing feature". In the other cases, the feature of importance is located within the individual diagram. For instance the feature of "squareness" is something that we can confirm or deny by looking at only a single figure. Similarly, the features of "brokenness" or "shaded" can be observed just by looking at a single shape. Of course knowing that the feature is a useful one to pick out to solve the problem requires looking at all of the choices. But this is fundamentally different from the feature of importance in this last answer, B, which is inherently relational. We cannot know about Figure A that it "has a distinguishing feature" until we check A, B, C, and D.

Associated with this relational aspect of the last solution is a sense of abstractness that is a very important concern of mathematics education. The other features that we have mentioned are concrete/material observables about the figure. Now it's true, they can be represented in propositional terms, "is a square" or "consists of broken lines". But "has a distinguishing feature" is inherently propositional. It cannot be represented in a physical or concrete modality.

Some very interesting recent work in the psychology of mathematics education helps us to better understand the mental processes associated with this sort of abstraction. Theories of inductive reasoning can help us to see how the concrete features of squareness or brokenness might be observed and selected, but they cannot help us to understand the processes of observing and selecting a propositional feature. Some other theory is needed. Such a theory has been developed under the name of reification promulgated by a number of mathematics education researchers, most notably Anna Sfard from Israel and her colleagues (Sfard 1995; Sfard & Linchevski 1994; Sfard & Thompson 1994). The key to understanding this cognitive leap is to notice that the propositional feature "has no distinguishing feature" comes about from reflection on the history of one's own engagement with the problem. Indeed, it is the process of examining the other solutions that itself becomes the object of further examination. It is only by repeatedly experiencing (in process) the distinguishing features in solutions A, C, and D, that one can formulate "has distinguishing feature" as a proposition (object).

This jumping of levels from process to object is what Sfard calls reification. It is deeply implicated in conceptual development in mathematics at all levels. For instance in arithmetic one may learn that the square root is a process of finding a certain number which when multiplied by itself gives a known number (e.g., $\sqrt{25} = 5$), but eventually one must understand $\sqrt{x}$ as an entity (object) in its own right. In fact, the process of square-rooting becomes a mental object. Sfard (1995) identifies reification in the progress of mathematics through its historical evolution, and in the developmental problems that so many students experience. Reification is more than the ability to name a process as an object. It is also the ability to refer back to the process when necessary. For instance variable symbols, x, y, z, etc. of algebra are only abstract representations to the extent that the student is able to relate them back to numerical processes when needed. Manipulating symbols without this melding of process and object is a kind of pseudo math (pretend math), that is very popular in my country (Sfard & Linchevski 1994).

Hiebert and Carpenter (1992) have proposed a general theory of conceptual understanding of mathematics involving two kinds of linkages between internal representations of ideas: "Networks [of linkages] may be structured like vertical hierarchies, or they may be structured like webs" (p. 67). The emphasis on horizontal (web-like) linkages generally is well understood by teachers who may try to link topics in the curriculum so that students see their interrelationships. Less well understood is the nature of vertical linkages that Sfard and others have begun to focus on. Such approaches show great promise for helping mathematics educators gain an integrated understanding of the mathematical knowledge of individual students.

The Social and Cultural Side of Cognition

As interesting, compelling, and useful as the preceding view of cognition is for mathematics education, it is important to observe that there are other aspects of mathematical knowledge and learning that have not been accounted for. Let us continue our exploration of the logic problem that we have been investigating.

One aspect of this problem that I believe is not adequately explained within a cognitive psychology framework is the tendency for most people to select A as their first choice rather than C or D. Notice that in a logical sense, the solutions A, C, and D are equivalent. Each involves identification of a physical feature that is unique to a particular figure. Why then do so many people choose A as their answer?

Apparently the preference concerns the salience of the feature "squareness" in comparison to the features of "brokenness" or "shaded". It is tempting to *naturalize* this greater distinctiveness of squareness —to believe that it is more distinctive in an *a priori* sense than these other features. But there is no empirical basis for such an assumption. Instead, I suggest, that the salience of squareness has to do with the prominence of the right angle in our culture's architecture, and the semiotic forms that have crystallized around it. Thus we teach properties of squares in school, examining their geometry in detail. I have no data to support this conjecture, but it would be interesting to perform a similar investigation with Eskimos whose architecture does not rely on straight lines or right angles, and whose survival is conditioned by the need to finely distinguish shades of snow color as an indication of the solidity of footing. Might we not expect them to choose C more often than A?

The work of Michael Cole (e.g., Cole 1977) and Sylvia Scribner (e.g., Scribner 1977) has been especially important for highlighting the social constitution of cognition. Notice the conflict between the assumption of social cognition and cognitive psychology which must produce a mechanical algorithm for the identification and selection of features. Now, of course, we could set up this algorithm to take account of social and cultural factors. And this approach is widely practiced as cognitive psychologists attempt to account for aspects of cognition that are more obviously social (e.g., Minsky 1975; Rumelhart 1980; Schank & Abelson 1977). But such efforts are inevitably post hoc. They can be used to describe social aspects of cognition, but never to explain the social constitution of cognition.

Continuing with our investigation, there is another aspect of the lem that cognitive psychology cannot account for. If you are like me, you value and appreciate the solution B for complex reasons. Part of the appeal of this solution is its abstract character which, as discussed before, is important for mathematics education. But the particular flavor of this abstraction is especially pleasing. It has an element of irony that appeals to lovers of paradox: The feature that distinguishes B from the other possible figures is that it has no distinguishing features. Such paradoxes are a staple of the foundations of mathematics (e.g., Russell's Paradox). But even if you are not familiar with Russell's Paradox, to the extent that you are mathematically enculturated, you will recognize this concern for contradictories as a central facet of mathematical thinking (e.g., the method of proof by contradiction).

This brings me to a principal concern of mathematics education for which cognitive psychology cannot provide insight or direction. Cognitive psychology is an expression of the classical dualism upon which modernism is based (Haugland 1980; Gardner 1987). As such it gains its power by factoring out the cognitive from the social (Anderson, Reder, & Simon 1997). Therefore curriculum, the cognitive content of mathematics, is factored out from the instructional practices through which curriculum is presented to students. The alternative, which I espouse, is to regard the function of education as enculturation, rather than as transmission of preset curricular knowledge. How we value paradox and contradiction, how we juxtapose generality and specificity, how we construct arguments and proofs are elements of mathematical culture that we cannot bring to our students through an exclusive focus on content. To be sure specific conceptual understandings are part of mathematical culture. But to accept cognitive psychology as the sole guiding light for education is to miss the valuable parts of mathematical culture that are not illuminated by it.

Fortunately, there are other lampposts that we can make use of in our educational missions. Vygotskyan *sociocultural theory* is concerned with the processes whereby novices appropriate competence through cultural activities (Leont'ev 1981).

> The objects in the child's world have a social history and functions that are not discovered through the child's unaided explorations. The function of a hammer, for example, is not understood by exploring the hammer itself (although the child may discover some facts about weight and balance). The child's appropriation of culturally devised "tools" comes about through involvement in culturally organized activities in which the tool plays a role. (Newman, Griffin & Cole 1989, pp. 62–63)

In Korea, I will have the privilege of collaborating on research into the *sociomathematical norms* (Yackel & Cobb 1996) of elementary school classrooms. Sociomathematical norms are aspects of the social organization of the classroom that are implicated in developing particularly mathematical ways of thinking and participating. In short, the social aspect of educational process is viewed as part of the curriculuar content of mathematics. This is the first step in related work in the United States that will form the basis for crosscultural comparisons. We look forward to contributing to some of the aspects of mathematics education that have not received sufficient attention in either of our countries because of the hegemony of individualist models of learning and knowing.

REFERENCES

Anderson, J. R. (1983): *The architecture of cognition*. Cambridge, MA: Harvard University Press.

Anderson, J. R.; Reder, L. M. & Simon, H. A. (1997): Situative versus cognitive perspectives: From versus substance. *Educational Researcher* **26(1)**, 18–21.

Beaton, A. E.; Mullis, V. S.; Martin, M. O.; Gonzalez, E. J.; Kelly, D. L. & Smith, T. A. (1996): Mathematics achievement in the middle school years: IEA's third international mathematics and science study (TIMSS). Chestnut Hill, MA: Boston College. http://wwwcsteep.be.edu/timss)

Cobb, P. (1994): Where is the mind? Constructivist and sociocultural perspectives on mathematical development. *Educational Researcher* **23(7)**, 13–20.

Cole, M. (1977): An ethnographic psychology of cognition. In: P. N. Johnson-Laird & P. C. Wason (Eds.), *Thinking: Readings in cognitive science* (pp. 468–482). Cambridge, UK:

Cambridge University Press.

Collins, A.; Brown, J. S. & Newman, S. (1989): Cognitive apprenticeship: Teaching the crafts of reading, writing, and mathematics. In: L.B. Resnick (Ed.), *Knowing, learning, and instruction* (pp. 453–493). Hillsdale, NJ: Lawrence Erlbaum Associates.

Evans, J. St. B. T. (1982): *The psychology of deductive reasoning.* London: Routledge & Kegan Paul.

Gardner, H. (1987): *The mind's new science* (2nd ed.). New York: Basic Books.

Greeno, J. G. (1993): For research to reform education and cognitive science. In: L. A. Penner, G. M. Batsche, H. M. Knoff, & D. L. Nelson (Eds.), *The challenges in mathematics and science education: Psychology's response* (pp. 153–192). Washington, DC: American Psychological Association.

Haugland, J. (1985): Semantic engines: An introduction to mind design. In: J. Haugland (Ed.), *Mind design* (pp. 1–34). Cambridge: The MIT Press.

Hiebert, J. & Carpenter, T. P. (1992): Learning and teaching with understanding. In: D. A. Grouws (Ed.), *Handbook of research in the teaching and learning of mathematics* (pp. 65–97). New York: Macmillan Co.

Hinsley, D. A., Hayes, J. R., & Simon, H. A. (1977): From words to equations: Meaning and representation in algebra word problems. In: M. Just & P. Carpenter (Eds.), *Cognitive processes in comprehension* (pp. 89–106). Hillsdale, NJ: Lawrence Erlbaum Associates.

Holland, J. H.; Holyoak, K. J.; Nisbett, R. E. & Thagard, P. R. (1989): *Induction.* Cambridge, Ma: The MIT Press.

Kirshner, D. & Whitson, J. (Eds.) (1997): *Situated cognition: Social, semiotic, and psychological perspectives.* Mahwah, NJ: Lawrence Erlbaum Associates.

Leont'ev, A. N. (1981): *Problems in the development of mind.* Moscow: Progress Publishers.

Minsky, M. (1975): A framework for representing knowledge. In: P. H. Winston (Ed.), *The psychology of computer vision.* New York: McGraw-Hill.

Newman, D.; Griffin, P. & Cole, M. (1989): *The construction zone.* Cambridge, UK: Cambridge University Press.

Paige, J. M. & Simon, H. A. (1966): Cognitive processes in solving algebra word problems. In: B. Kleinmuntz (Ed.), *Problem solving.* New York: John Wiley & Sons, Inc.

Rumelhart, D. (1980): Schemata: The building blocks of cognition. In: R. J. Spiro, B. C. Bruce, & W. F. Brewer (Eds.), *Theoretical issues in reading and comprehension* (pp. 33–58). Hillsdale, NJ: Lawrence Erlbaum Associates.

Schank, R. C., & Abelson, R. (1977): *Scripts, plans, goals, and understanding.* Hillsdale, NJ: Lawrence Erlbaum Associates.

Scribner, S. (1977): Modes of thinking and ways of speaking: Culture and logic reconsidered. In: P. N. Johnson-Laird & P. C. Wason (Eds.), *Thinking: Readings in cognitive science* (pp. 483–500). Cambridge, UK: Cambridge University Press.

Sfard, A. (1995): The development of algebra: Confronting historical and psychological perspectives. *Journal of Mathematical Behavior* **14(1)**, 15–39.

Sfard, A. & Linchevski, L. (1994): The gains and the pitfalls of reification: The case of algebra. *Educational Studies in Mathematics* **26**, 191–228.

Sfard, A. & Thompson, P. W. (1994): Problems of reification: Representations and mathematical objects. In: D. Kirshner (Ed.), *Proceedings of the Sixteenth Annual Meeting of the International Group for the Psychology of Mathematics Education, North American Chapter*, Vol. 1 (pp. 3–34).

Yackel, E., & Cobb, P. (1996): Sociomathematical norms, argumentation, and autonomy in mathematics. *Journal for Research in Mathematics Education* **27(4)**, 458–477.