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Design and Estimation of Double Sampling Plans for the Dependent Production Processes

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Abstract

In this paper, design procedure and estimation of the double sampling plans are developed when the production process is examined in order and if it shows the dependence between the products. If a dependent process model can be simulated, the best sampling plans can be selected by using the special properties of the probability structure. The number of actual evaluations to find the plans can be reduced remarkably. The experimental study reveals that only small portion of the total exhaustive enumeration is needed. ARMA (1,1) time series models are given as numerical examples.

1. Introduction

In standard acceptance sampling plans, statistical sampling procedures are based on independent and identically distributed random variables. In practical use, it may be difficult to attain a state of statistical control with this strict sense; dependence or correlation between items and other systematic time related effects are sometimes substantial.

Broadbent[4] observed a model in which the output of a process produced good and bad items. He investigated a production process where a mold was continuously producing glass products(TV tube) in an automatic manufacturing scheme. He found that Markovian depen-

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dence was introduced by the fact that a defect was likely to occur in a succession of items from a single mold until the mold was renewed. The dependence was unavoidable because of the gradual quality degradation of the mold. The items produced from the next shaping process was necessarily affected by the items produced from the molding process. Johnson and Bagshaw[9] studied the effects of serial correlation in a continuous sheet-like process on the performance of a one-sided cusum test proposed by Page[13]. The process was desired to control the weight at 1.25 per unit area. But the deviations from the target were identified as a time series AR(1) model with $\phi=0.65$. Due to this serial correlation, not only the average run length was increased but, more importantly, the run length distribution itself was changed. Cox[6] studied an acceptance sampling situation in which the proportion of defective items in the batches formed two state Markov chain with known transition probabilities. Campling[5] investigated serial sampling acceptance schemes for large batches of items when the fraction of defective ratio parameter formed continuous state Markov process, and the number of defective items followed a normal distribution. Preston[16] studied an empirical Bayes estimation problem for the application of a single acceptance sampling in which the set of parameter values was a realization of a stationary Markov chain. Kumar[10] has developed a tightened m-level continuous sampling plan that was an extension of MLP-T plans for s independence developed by Derman, Littauer, and Solomon[7] to the Markov-dependent production processes. Sarkadi and Vincze[18] analyzed a single sampling plan on a Polya sequence model. Alwan and Roberts[1] used two basic charts called common-cause chart and Special-cause chart when the data shows lack of statistical control. As another effort to avoid the dependence, Skip lot plans(Perry[14]) have been usually used if there are patterned variations of dependent incoming lot quality. An example of this would be successive lots of product coming from a process that exhibits deterioration over time due to such factors as tool wear. In such situations, sampling inspections are concentrating on the portion of the incoming lots exhibiting the poorest quality. Bhat, Lal and Karunaratne[3] approached the single acceptance sampling problem with an augmented Markov chain matrix. Nelson[11] has developed a method for estimating single acceptance sampling plans for general production process models using simulation.

An Example of Inspection Problems in Dependent Production Process

Until recently, in many companies there has been little attempt to control the quality of a product at each stage of production, nor has there been attempt to determine what factors influence product quality. The quality of an item has been determined by the time it reaches the final inspection stage. Thus 100% final inspection, as has been routinely practiced in many companies both large and small, will not ensure good quality. Studies have shown that only about 80% of nonconforming units are detected during 100% final inspection (Ryan [17]). It has been reported that in many cases 100% inspection tasks are not error free, but on the contrary may even be error prone due to the inspection error(Shin and Lingayat[20]). It is desirable to collect sample data at every stage of a production process and analyze those data. Here statistical methods such as control charts or acceptance samplings are used at each stage of production to inspect the quality, and other statistical techniques are used during the production. In a series of production facilities, product quality of the previous production stage necessarily affects the quality of the next production stage. Suppose, for example, a new tool will be used at the beginning of process and during the production progressive tool wear will take place. The probability of producing a defective item will vary as items are produced. Assume that $\{Y_i, i=1,2,\cdots,N\}$ are the measurements of a production process. If the process measurements follow ARMA(1,1) process, the series is represented as

$$Y_i = \mu + \phi(Y_{i-1} - \mu) + \theta\,\varepsilon_{i-1} + \varepsilon_i, \text{ for } i = 1, 2, \cdots, N,$$

where μ is the mean of the process, and $\{\varepsilon_i, i=1,2,\cdots,N\}$ follows $N(0,\sigma^2)$, $|\phi| \le 1$, and $|\theta| \le 1$. The process measurement Y_i also follows a normal distribution $N(\mu, \sigma_Y^2)$, where $\sigma_Y^2 = \sigma^2(1+\theta^2+2\phi\theta)/(1-\phi^2)$.

The two parameters ϕ and θ characterize the process values. If $\phi=\theta=0$, the process corresponds to the independent process. If the ith item is within tolerance limits, that is $l \le Y_i \le u$, the item is accepted, otherwise the item is rejected. The degradation in quality can be due to a shift in the process mean or increase of process variance. The fraction of defectives p, is dependent on μ and σ_Y^2. For examples, two ARMA(1,1) models with the following process specifications have been considered by Nelson[11].

	increasing variance		shifted mean	
Quality	AQL	LTPD	AQL	LTPD
μ	10	10	10	11.2940
σ_Y^2	0.03778	0.09240	1	1
lower limit	9.5	9.5	7.4242	7.4242
upper limit	10.5	10.5	12.5758	12.5758
p	0.01	0.10	0.01	0.10

Figure 1 is an example graph of the increasing variance case showing typical realizations of the process value Y_i. In the increasing variance case, the mean μ is fixed and the degradation in quality is due to an increase of process variance σ_Y^2. In the shifted mean case, variance σ_Y^2 is fixed and a shift of the mean μ degrades the quality. With the acceptable tool wear the producer's risk is α at the prespecified AQL, and with more rapid tool wear the consumer's risk is β at the

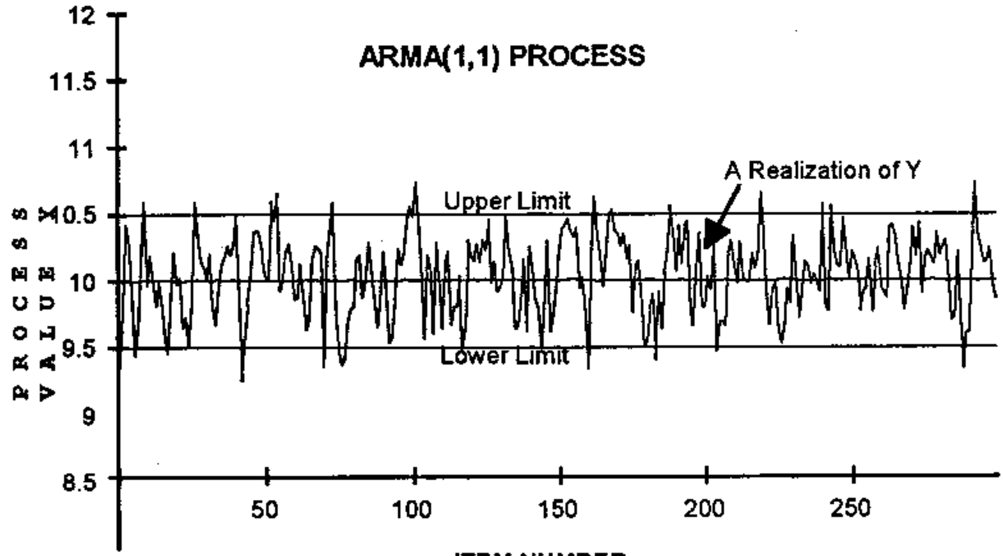


Figure 1. A typical realization of ARMA(1,1) process value Y_i

specified LTPD. Suppose products of lot size N are inspected sequentially as they are produced. When dependent items are produced due to such as tool wear, the process is stopped and rearranged. Here, acceptance sampling plans are used for determining the process stopping or rejection of the lot. However, we can not use conventional sampling plans since it is based on independent observations. The purpose of this paper is the discussion of the design procedure for the double sampling plans in such dependent production processes.

At the next section, we will find rejection probabilities of a lot to design sampling plans. The design procedure of double sampling plans and search schemes will be discussed in section 3 and 4. Section 5 describes the measure of performances of the plans. Numerical results of the above example will be shown at the last section.

2. Estimation of a Rejection Probability

Let consider a production process having a lot of size N. We assume that the production has the stochastic output process $\{X_1, X_2, \cdots, X_N\}$. If the state of the process producing the ith item is good, for example, in the above example if $l \le Y_i \le u$, then $X_i=0$, otherwise $X_i=1$. The inspector tests each item sequentially until a decision is made according to the constructed sampling plan. Let $C_i = \sum_{k=1}^{i} X_k$ be the cumulative number of defective items discovered through item i for i=1, 2, ..., N and let $\gamma_{ij} = Pr(C_i \ge j)$ be the rejection probability of a lot under a production process. By applying the method of Monte-Carlo integration we can get the estimator of γ_{ij} :

$$\hat{\gamma}_{ij}=\sum_{k=1}^{m} I(C_i \geq j)/m,$$

where $I(C_i \geq j)$ is 1 if $C_i \geq j$ and 0 otherwise, and m is the total number of simulation replications. Denote $S_{ij}=\sum_{k=1}^{m} I(C_i \geq j)$, which represents the number of replications that j or more defective items are found among i items. Then, the estimator $\hat{\gamma}_{ij}$ is $S_{ij}/m=\overline{S_{ij}}$.

Now, let us define $f_{ij}=Pr(C_i=j)$ be the probability that there are exactly j defective items among the i items, and define D_j to be the order of inspected item on which the jth defective item found. The event $\{D_j=i\}$ means that the jth defective item occurs at the ith item and let the probability of this event is v_{ij}. Then the following two events A and B are equivalent.

A= {There are j or more defective items among the first i items} = $\{C_i \geq j\}$

B= {The jth defective item is found at or before the ith item} = $\{D_j \leq i\}$

Theorem 1 : Note that $\gamma_{i0}=1$ and $\gamma_{i,i+1}=0$. For all $i=1, 2, \cdots, N$ and $j=1, 2, \cdots, i$,

(1) $v_{ij}=\gamma_{ij}-\gamma_{i-1,j}$ (2) $f_{ij}=\gamma_{ij}-\gamma_{i,j+1}$

Proof : Since the two events A and B are equivalent, the probabilities of the two events are also same. Thus,

$$\gamma_{ij}=\Pr(C_i \geq j)=\Pr(D_j \leq i)=\sum_{k=1}^{j-1}\Pr(D_j=k) + \sum_{k=j}^{i}\Pr(D_j=k)=\sum_{k=j}^{i} v_{kj}.$$

This gives that $\gamma_{ij}-\gamma_{i-1,j}=v_{ij}$, and $\gamma_{ij}-\gamma_{i,j+1}=f_{ij}$.

Theorem 2 : The rejection probability γ_{ij} has the following two monotonous properties for $i=1, 2, \cdots, N$ and $j=1, 2, \cdots, i$.

(1) Column monotonous property of γ_{ij} : For fixed j, γ_{ij} is non -decreasing when i increases, i.e., $\gamma_{kj} \geq \gamma_{i-1,j}$ for $k=i, i+1, \cdots, N$.

(2) Row monotonous property of γ_{ij} : For fixed i, γ_{ij} is non-increasing when j increases, i.e., $\gamma_{ij} \geq \gamma_{ik}$ for $k=j+1, j+2, \cdots, i$.

Proof :

(1) Since $v_{ij}=\gamma_{ij}-\gamma_{i-1,j} \geq 0$, we get $\gamma_{ij} \geq \gamma_{i-1,j}$.

(2) Since $f_{ij}=\gamma_{ij}-\gamma_{i,j+1} \geq 0$, we get $\gamma_{ij} \geq \gamma_{i,j+1}$.

3. Design of Double Sampling Plans

Let us define the decision variables n_i, a_i, r_i be the sample size, acceptance number, and rejection number of the ith sampling stage for i=1,2 respectively. The goal of double sampling plan is to find the 5 variables $(n_1, a_1, r_1, n_2, r_2)$ since $a_2=r_2-1$ under producer's risk α at AQL and consumer's risk β at LTPD. Probabilities of the four events

A_1= {Lot is accepted at the first stage},

A_2= {Lot is accepted at the second stage},

R_1= {Lot is rejected at the first stage}, and

R_2= {Lot is rejected at the second stage} are respectively

$$\Pr\{A_1\}=\Pr(C_{n_1} \leq a_1)=1-\gamma_{n_1,a_1+1},$$

$$\Pr\{R_1\}=\Pr\{C_{n_1} \geq r_1\}=\gamma_{n_1,r_1},$$

$$\Pr\{A_2\}=\sum_{k=a_1+1}^{r_1-1}\Pr(C_{n_1}=k)\Pr(C_{n_2}<r_2-k)$$

$= \sum_{k=a_1+1}^{r_1-1} f_{n_1,k}(1-\gamma_{n_2,r_2-k})$, and

$\Pr\{R_2\} = \sum_{k=a_1+1}^{r_1-1} \Pr(C_{n_1}=k)\Pr(C_{n_2} \geq r_2-k)$

$= \sum_{k=a_1+1}^{r_1-1} f_{n_1,k}\gamma_{n_2,r_2-k}.$

The rejection probability of a lot is

$\gamma(n_1, a_1, r_1, n_2, r_2) = \Pr\{R_1\} + \Pr\{R_2\}$

$= \gamma_{n_1,r_1} + \sum_{k=a_1+1}^{r_1-1} f_{n_1,k}\gamma_{n_2,r_2-k}.$

Theorem 3 :

Monotonous properties of the rejection probability $\gamma(n_1, a_1, r_1, n_2, r_2)$

(1) Column monotonous property : For given fixed n_1, a_1, r_1 and r_2, $\gamma(n_1, a_1, r_1, n_2, r_2)$ is non-decreasing when n_2 increases, i.e.,

$\gamma(n_1, a_1, r_1, n_2, r_2) \geq \gamma(n_1, a_1, r_1, n_2-1, r_2).$

(2) Row monotonous property : For given fixed n_1, a_1, r_1 and r_2, $\gamma(n_1, a_1, r_1, n_2, r_2)$ is non-increasing when r_2 increases, i.e.,

$\gamma(n_1, a_1, r_1, n_2, r_2+1) \leq \gamma(n_1, a_1, r_1, n_2, r_2).$

(3) Acceptance number monotonous property : For given fixed n_1 and r_1, $\gamma(n_1, a_1, r_1, n_2, r_2)$ is non-increasing when a_1 increases, i.e.,

$\gamma(n_1, a_1, r_1, n_2, r_2) \geq \gamma(n_1, a_1+1, r_1, n_2, r_2).$

Proof :

(1) $\gamma(n_1, a_1, r_1, n_2, r_2) - \gamma(n_1, a_1, r_1, n_2-1, r_2)$

$= \sum_{k=a_1+1}^{r_1-1} f_{n_1,k}(\gamma_{n_2,r_2-k} - \gamma_{n_2-1,r_2-k})$

$= \sum_{k=a_1+1}^{r_1-1} f_{n_1,k} v_{n_2,r_2-k} \geq 0$

(2) $\gamma(n_1, a_1, r_1, n_2, r_2) - \gamma(n_1, a_1, r_1, n_2, r_2+1)$

$= \sum_{k=a_1+1}^{r_1-1} f_{n_1,k}(\gamma_{n_2,r_2-k} - \gamma_{n_2,r_2+1-k})$

$= \sum_{k=a_1+1}^{r_1-1} f_{n_1,k} f_{n_2,r_2-k} \geq 0$

(3) $\gamma(n_1, a_1, r_1, n_2, r_2)$

$= \gamma_{n_1,r_1} + \sum_{k=a_1+1}^{r_1-1} f_{n_1,k}\gamma_{n_2,r_2-k}$

$\geq \gamma_{n_1,r_1} + \sum_{k=a_1+2}^{r_1-1} f_{n_1,k}\gamma_{n_2,r_2-k}$

$= \gamma(n_1, a_1+1, r_1, n_2, r_2)$

Lemma 1 : We can regard that

$\hat{\gamma}(n_1, a_1, r_1, n_2, r_2)$

$= \overline{S_{n_1,r_1}} + \sum_{k=a_1+1}^{r_1-1} (\overline{S_{n_1,k}} - \overline{S_{n_1,k+1}})\overline{S_{n_2,r_2-k}}$

is an unbiased estimator of $\gamma(n_1, a_1, r_1, n_2, r_2)$, i.e.,

$E[\hat{\gamma}(n_1, a_1, r_1, n_2, r_2)] \approx \gamma(n_1, a_1, r_1, n_2, r_2).$ ∎

Proof : Refer to the Appendix.

Since a double sampling plan has five decision variables there are considerable varieties of sampling plans. Let us denote $\beta^* = 1-\beta$. The sampling plan has the following two constraints :

(1) *α-constraint* :

$\hat{\gamma}(n_1, a_1, r_1, n_2, r_2; \text{AQL}) = \hat{\alpha}_1 + \hat{\alpha}_2 \leq \alpha$

(2) *β-constraint* :

$\hat{\gamma}(n_1, a_1, r_1, n_2, r_2; \text{LTPD}) = \hat{\beta}_1^* + \hat{\beta}_2^* \geq \beta^*,$

where $\hat{\alpha}_i$ and $\hat{\beta}_i^*$, i=1,2, are probabilities that a lot is rejected during the ith stage at AQL and LTPD respectively.

Nelson[12] has suggested three selection criteria such as minimum-n, nearest-α, minimum-loss plans in single sampling plans (n, c), where c is the maximum allowable number of defects in n samples. Minimum-n criterion selects a plan if we are interested in minimizing the sample size satisfying α and β-constraint. Nearest-α criterion selects a plan minimizing $\alpha - \overline{S_{ij}}$ over all feasible plans; that is, to find the feasible plan that comes nearest to the specified producers risk. Here, as one of reasonable selection criteria, we will use minimum loss plan that minimizes loss function. The loss function is defined as :

$$L(n_1, a_1, r_1, n_2, r_2) = | \hat{\gamma}(n_1, a_1, r_1, n_2, r_2; AQL) - \alpha | + | \hat{\gamma}(n_1, a_1, r_1, n_2, r_2; LTPD) - \beta^* |$$

Finding the combination that minimizes the loss function requires searching through all possible combinations of variables, that is, the search scheme must form five-level nested iteration loops. The efficiency of the search scheme is dependent on the search order and the ranges of variables. The feasible ranges of decision variables are as follows.

Variable	Range
n_1	$2, \cdots, N/2$
r_1	$2, \cdots, n_1$
a_1	$0, \cdots, r_1-2$
n_2	$n_1, \cdots, N-n_1$
r_2	$r_1, \cdots, r_1-1+n_2$

The variables will be decided in the order from n_1 to r_2 as shown above. Practically, however, the exhaustive enumeration of the loss function for all the decision variables is impossible because the total number of possible combinations is very large if lot size N is fairly large. The Total Loop Count (TLC) of the possible combinations is

$$TLC = \sum_{n_1=2}^{N/2} \sum_{r_1=2}^{n_1} \sum_{a_1=0}^{r_1-2} \sum_{n_2=n_1}^{N-n_1} \sum_{r_2=r_1}^{r_1-1+n_2} 1.$$

For example, the following shows that TLC increases polynomially when N increases.

N	TLC
100	26,031,250
150	197,718,750
200	833,250,000
250	2,542,968,750
300	6,327,843,750

If properties of $[\overline{S}_{ij}]$ are properly used and the variable ranges are well restricted by using the relationships between them, the TLC can be reduced. We will consider this at the next section.

4. Search Schemes

If the minimum and maximum values of $\hat{\alpha}$ and $\hat{\beta}^*$ can be determined in advance before the execution of all loop variable, the execution of unnecessary iterations can be omitted. Only when the observed intervals (min $\hat{\alpha}$, max $\hat{\alpha}$) and (min $\hat{\beta}^*$, max $\hat{\beta}^*$) include true parameters α and $\hat{\beta}^*$ respectively, the next inner loop will be executed. We will call such an interval as the *feasible interval*. The narrower the feasible interval, the more reduction of the unnecessary iterations can be obtained. The iterations continue until the decision variable exceeds its final value. If both α and β^* intervals include a feasible decision variable, then the next inner loop will be executed. At the inner loop, since one additional decision variable is known, more information for finding the minimum and maximum of α and β^* is available. This, in turn, means that the searching interval range becomes narrower. This procedure is repeated until the most inner loop, that is, r_2 loop is reached. At this loop level, since four decision variables n_1, a_1, r_1, n_2 are fixed, the narrowest feasible interval range is obtained. Now the loss function can be evaluated for each r_2 variable. We will call this search scheme as *loop screen search scheme* (LSSS). To find the minimum and maximum of $\hat{\alpha}$ and n_1 at each loop level efficiently, in other words to find the narrowest feasible intervals, define the variables min $\hat{\alpha}(n_1, a_1, r_1, n_2)$, min $\hat{\alpha}(n_1, a_1, r_1)$, min $\hat{\alpha}(n_1, a_1)$, and min $\hat{\alpha}(n_1)$ are minimum values of $\hat{\gamma}(n_1, a_1, r_1, n_2, r_2;AQL)$ when the variables in the parenthesis are fixed respectively. In a similar way, define max $\hat{\alpha}(n_1, a_1, r_1, n_2)$, max $\hat{\alpha}(n_1, a_1, r_1)$, max $\hat{\alpha}(n_1, a_1)$, and max $\hat{\alpha}(n_1)$ as the maximum values of $\hat{\gamma}(n_1, a_1, r_1, n_2, r_2;\text{AQL})$ as above. The interval limits of $\hat{\alpha}$ at each loop level can be found from the following proposition 1. The minimum and maximum of $\hat{\beta}^*$ are defined and found exactly the same way as $\hat{\alpha}$ case at LTPD.

Proposition 1 : The interval limits for the variable screening are as follows.

(1) The screen of r_2 loop :

min $\hat{\alpha}(n_1, a_1, r_1, n_2)$=

$\overline{S_{n_1,r_1}} + \sum_{k=a_1+1}^{r_1-1}(\overline{S_{n_1,k}} - \overline{S_{n_1,k+1}})\overline{S_{n_2,r_1-1+n_2-k}}$, and

max $\hat{\alpha}(n_1, a_1, r_1, n_2)$=

$\overline{S_{n_1,r_1}} + \sum_{k=a_1+1}^{r_1-1}(\overline{S_{n_1,k}} - \overline{S_{n_1,k+1}})\overline{S_{n_2,r_1-k}}$.

(2) The screen of n_2 loop :

min $\hat{\alpha}(n_1, a_1, r_1)$=

$\overline{S_{n_1,r_1}} + \sum_{k=a_1+1}^{r_1-1}(\overline{S_{n_1,k}} - \overline{S_{n_1,k+1}})\overline{S_{n_1,r_1-1+n_1-k}}$, and

max $\hat{\alpha}(n_1, a_1, r_1)$=

$\overline{S_{n_1,r_1}} + \sum_{k=a_1+1}^{r_1-1}(\overline{S_{n_1,k}} - \overline{S_{n_1,k+1}})\overline{S_{N-n_1,r_1-k}}$.

(3) The screen of a_1 loop :

min $\hat{\alpha}(n_1, a_1)$=

$\overline{S_{n_1,r_1}} + (\overline{S_{n_1,r_1-1}} - \overline{S_{n_1,r_1}})\overline{S_{n_1,n_1}}$, and

max $\hat{\alpha}(n_1, a_1)$=

$\overline{S_{n_1,r_1}} + \sum_{k=1}^{r_1-1}(\overline{S_{n_1,k}} - \overline{S_{n_1,k+1}})\overline{S_{N-n_1,r_1-k}}$.

(4) The screen of r_1 loop :

min $\hat{\alpha}(n_1) = \overline{S_{r_1,r_1}} + (\overline{S_{r_1,r_1-1}} - \overline{S_{r_1,r_1}})\overline{S_{r_1,r_1}}$, and

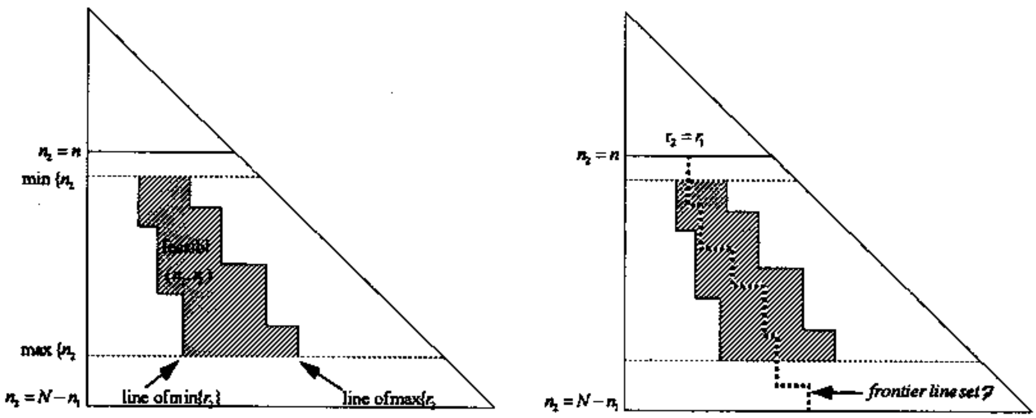
max $\hat{\alpha}(n_1) = \overline{S_{n_1,2}} + (\overline{S_{n_1,1}} - \overline{S_{n_1,2}})\overline{S_{N-n_1,1}}$.

For the most outer n_1 loop, we can not decide the minimum and maximum values because min $\hat{\alpha}(n_1)$ and max $\hat{\alpha}(n_1)$ are not monotonous functions of n_1.

Proof : Refer to the Appendix.

Now, we will discuss *frontier line search scheme*(FLSS) to find variables n_2 and r_2 faster. Since $n_2 \geq n_1$ and $r_2 \geq r_1$, begin n_2 from n_1 and r_2 from r_1.

If $\hat{\gamma}(n_1, a_1, r_1, n_2, r_2;\text{AQL}) < \alpha$, the larger n_2 is required in order to increase the value of $\hat{\gamma}(n_1, a_1, r_1, n_2; r_2;\text{AQL})$ by the column monotonous property. The increased n_2 reduces $|\hat{\gamma}(n_1, a_1, r_1, n_2, r_2;\text{AQL}) - \alpha|(=L_1)$. Continuously increase only n_2 until α-constraint is violated. Then L_1 is minimized at this point. If n_2 increases more, it increases $\hat{\gamma}(n_1, a_1, r_1, n_2, r_2;\text{AQL})$ and hence degrades the loss function. So, if $\hat{\gamma}(n_1, a_1, r_1, n_2, r_2;\text{AQL}) > \alpha$, the larger r_2



(a) LSSS (b) FLSS

Figure 2. The feasible area of (n_2, r_2) in LSSS and the frontier line set F in FLSS

is required in order to decrease the value of $\hat{\gamma}(n_1, a_1, r_1, n_2, r_2;\text{AQL})$ by the row monotonous property. Variable r_2 increases until it reaches maximum value r_1-1+n_2 and at this time n_2 is increased by 1. The procedure is repeated until n_2 exceeds N-n_1. The screening of variables (n_2, r_2) is finished and the next screening loop of a_1 is resumed. The searched point (n_2, r_2) forms a *frontier line*, which is a set satisfying the following property.

Property : For fixed n_1, a_1 and r_1, there exists a *frontier line* set $F=\{(i_2, j_2)$ such that if $\hat{\gamma}(n_1, a_1, r_1, i_2, j_2;AQL) \leq \alpha$, then $\hat{\gamma}(n_1, a_1, r_1, i, j\ ;AQL) > \alpha$ for $i=i_2+1$ or $j=j_2-1\}$.

In other words, FLSS is a scheme that searches the best decision variables (n_2, $r2$) when n1 ,a1 and r1 are fixed along the *frontier line* set F. Figure 2 shows an example of the feasible point set (n_2, r_2) in LSSS and frontier line set F in FLSS.

5. Measure of Performances for Dependent Sampling Plans

The formula of measure of performance for independent case directly can not be used because the fraction of defective is not constant in dependent model. See Schilling[19] for the formula of independent model. In case of semi curtailed inspection, all n_1 items of the first stage are inspected fully and curtailment occurs only at the second stage. Under this condition, average sample number(ASN) is

$$\text{ASN}=n_1+\sum_{k=a_1+1}^{r_1-1}[n_2\Pr(C_{n_2}\leq r_2\text{-}1\text{-}k) + \sum_{l=1}^{n_2} l Pr(D_{r_2}=l+n_1)]\Pr(C_{n_1}=k).$$

Since average total inspection(ATI) is not affected by the curtailment strategy, from the definition of ATI we have

$$ATI=n_1\Pr\{A_1\}+(n_1+n_2)\Pr\{A_2\}+N(\Pr\{R_1\}+\Pr\{R_2\}).$$

To find the average outgoing quality(AOQ) we need to know the average number of defective items remaining in a lot and average number of items actually shipped after inspection. We can obtain the expectation of C_n as follow.

$$E[C_n]=\sum_{j=1}^{\infty}\Pr\{C_n\geq j\}=\sum_{j=1}^{\infty}\gamma_{nj}=\sum_{j=1}^{n}\gamma_{nj}$$

Let $[N_D]_i^j$ represent the number of defective items among the item number i through j, where j>i. Obviously, $[N_D]_i^j=[N_D]_1^j-[N_D]_1^{i-1}$. Since $[N_D]_1^n$ is simply C_n, the expected value of $[N_D]_i^j$ is

$$E\{[N_D]_i^j\}=E\{[N_D]_1^j\}-E\{[N_D]_1^{i-1}\}$$
$$=E[C_j]-E[C_{i-1}]=\sum_{k=1}^{j}\gamma_{jk}-\sum_{k=1}^{i-1}\gamma_{i-1,k}.$$

Let N_D be the total number of defective items remaining in a lot of size N after inspection. Also let $N_D|R_k$ and $N_D|A_k$ are the number of defective items remaining in a lot of size N under the event R_k and A_k for k=1,2 respectively. Then the expected value of N_D is

$$E[N_D]=E[N_D|R_1]\Pr\{R_1\}+E[N_D|R_2]\Pr\{R_2\}+$$
$$E[N_D|A_1]\Pr\{A_1\}+E[N_D|A_2]\Pr\{A_2\}$$
$$=E[N_D|A_1]\Pr\{A_1\}+E[N_D|A_2]\Pr\{A_2\},$$

where

$$E[N_D|A_1]=E\{[N_D]_{n_1+1}^N\}=\sum_{j=1}^{N}\gamma_{Nj}-\sum_{j=1}^{n_1}\gamma_{n_1,j},\ \text{and}$$
$$E[N_D|A_2]=E\{[N_D]_{n_1+n_2+1}^N\}=\sum_{j=1}^{N}\gamma_{Nj}-\sum_{j=1}^{n_1+n_2}\gamma_{n_1+n_2,j}.$$

Let N_S denote the number of items actually shipped after inspection. If all defective items found are replaced with good ones, then $N_s=N$. When the defective items found are discarded and not replaced with good ones, $E[N_s]$ is calculated as follow.

$$E[N_s]=(N-E\{[N_D]_1^{n_1}\})[\Pr\{A_1\}+\Pr\{R_1\}]$$
$$+(N-E\{[N_D]_{n_1+n_2+1}^N\})\Pr\{A_2\}+(N-r_2)\Pr\{R_2\}.$$

Now AOQ can be found from the definition AOQ= $E[N_D]/E[N_s]$.

6. Implementation of Double Sampling Plans and A Numerical Example

For numerical examples, we will consider ARMA (1,1) time series model introduced earlier with lot size N=300, α=0.1, β=0.1, AQL=0.01, LTPD=0.1 and number of replications m=10000. To illustrate and compare the effect of the dependence, three different cases are considered.

(1) independent case (ϕ,θ)=(0, 0)

(2) dependent case showing diminishing correlation for farther apart items in sequence, (ϕ,θ)= (0.25, 0.25)

(3) more dependence case, (ϕ,θ)=(0.5, 0.25)

The process measurements are dependent with lag correlation :

$$\rho_k=\phi^{k-1}(1+\phi\theta)(\phi+\theta)/(1+\theta^2+2\phi\theta),\ k=1,2,3,\cdots.$$

(ϕ,θ)	(0,0)	(0.25,0.25)	(0.5,0.25)
ρ_k	0	$0.45(0.25)^{k-1}$	$0.64(0.5)^{k-1}$

Many selection criteria have been suggested (Pfanzagl[15]) for selecting double sampling plans. Usually, trial and error procedures have been attempted widely (Guenther[8], Olorunniwo and Salas[12]). As a selection criterion, the following restricted plans will be considered and compared with each other.

Plan	Restrictions
D1	None
D2	$n_2 = n_1$
D3	$n_2 = 2n_1$
D4	$r_1 = r_2$
D5	$n_2 = 2n_1$, $r_1 = r_2$

Table 1 shows the efficiency of LSSS and FLSS in finding double sampling plans for increasing variance case when (ϕ, θ)=(0.25, 0.25). The summaries of the simulation results of the increasing variance case are shown in Table 2 and Table 3, and the results of the shifted mean case are shown in Table 4 and Table 5 including single sampling plan to compare with other plans. Simulation was performed on an IBM PC with Pentium 133MHz processor.

Table 1. Efficiency of search schemes for increasing variance, (ϕ, θ)=(0.25,0.25)

	n_1	r_1	a_1	TLC
total	149	10965	533	6,327,843,750
screened	129	189	230	44052
screened (%)	86.6%	1.7%	44.1%	0 .000696%

In Table 1, the row "total" and "screened" represent number of loop counts before and after applying search schemes respectively. Among the 149 iterations of n_1 loop, 129 iterations(86.6%) have feasible intervals. For loop r_1 and a_1 loops, 1.7% and 44.1 % have feasible intervals respectively. The column "TLC" shows the total iterations of the loss function. Among the exhaustive TLC=6,327,843,750 iterations, only 44,052(0.000696%) iterations are executed, which shows that the LSSS and FLSS are very efficient algorithms by reducing the unnecessary iterations remarkably.

In Table 2 and 4, $\hat{\alpha}$ and $\hat{\beta}^*(=1-\beta)$ represent the observed rejection probabilities at AQL and LTPD respectively. As expected, plan D1(no restriction) selected a minimum loss plan. Some plan got the same results of other plan coincidentally, for example, D3 and D5, D1 and D4 when (ϕ, θ)=(0, 0) and (0.25, 0.25). Comparing the loss, we can see that plan D2(n_2=n_1 restriction) has stricter restriction than plan D4(r_2=r_1 restriction). In general, plan D3 (n_2=2n_1 restriction) having the same or similar result of D1 seems more reasonable than D2 or D4. Plan D5(the strictest restrictions, n_2=2n_1, r_2=r_1) resulted in larger loss than others. The less precision of $\hat{\alpha}$ or larger loss in a single plan shows that double sampling plans select more elaborate plans. The most outstanding feature is that the larger dependence resulted in the larger sample size or decision variables of both single and double sampling plans.

Table 3 and 5 show the measure of performances of sampling plans. The ASN is smaller than that of single sampling plan at AQL, which is one of the advantage of the

Table 2. The selected sampling plans for increasing variance

(ϕ,θ)	plan	n_1	a_1	r_1	n_2	r_2	$\hat{\alpha}$	$\hat{\beta}^*$	loss
(0,0)	single	38		2			0.057	0.901	4.37E-02
	D1	23	0	2	51	2	0.100	0.904	4.54E-03
	D2	24	0	2	24	2	0.069	0.896	3.52E-02
	D3	23	0	2	46	2	0.094	0.903	8.99E-03
	D4	23	0	2	51	2	0.100	0.904	4.54E-03
	D5	23	0	2	46	2	0.094	0.903	8.99E-03
(0.25,0.25)	single	42		2			0.073	0.902	2.87E-02
	D1	26	0	2	49	2	0.100	0.903	2.82E-03
	D2	28	0	2	28	2	0.084	0.907	2.31E-02
	D3	25	0	2	50	2	0.097	0.893	1.05E-02
	D4	26	0	2	49	2	0.100	0.903	2.82E-03
	D5	25	0	2	50	2	0.097	0.893	1.05E-02
(0.5,0.25)	single	48		2			0.105	0.899	6.10E-03
	D1	32	0	2	88	3	0.100	0.902	1.54E-03
	D2	33	0	2	33	2	0.105	0.898	6.98E-03
	D3	33	0	2	66	3	0.091	0.906	1.44E-02
	D4	32	0	3	122	3	0.100	0.903	2.86E-03
	D5	49	1	3	98	3	0.074	0.904	2.94E-02

Table 3. Measure of performances for increasing variance

(ϕ,θ)	plan	at AQL				at LTPD			
		ASN cut	AOQ no replace	AOQ replace	ATI	ASN cut	AOQ no replace	AOQ replace	ATI
(0,0)	single	37.2	0.0084	0.0084	53.0	19.0	0.0087	0.0087	274.0
	D1	29.5	0.0083	0.0082	56.5	23.9	0.0090	0.0088	273.5
	D2	27.8	0.0086	0.0086	46.6	24.9	0.0096	0.0094	271.7
	D3	29.0	0.0084	0.0083	54.6	24.0	0.0090	0.0089	273.3
	D4	29.5	0.0083	0.0082	56.5	23.9	0.0090	0.0088	273.5
	D5	29.0	0.0084	0.0083	54.6	24.0	0.0090	0.0089	273.3
(0.25,0.25)	single	40.8	0.0080	0.0080	60.9	20.3	0.0085	0.0084	274.7
	D1	32.2	0.0081	0.0081	58.9	26.9	0.0089	0.0088	273.5
	D2	32.3	0.0082	0.0082	54.8	28.8	0.0084	0.0083	275.0
	D3	31.1	0.0081	0.0081	57.1	26.0	0.0099	0.0098	270.6
	D4	32.2	0.0081	0.0080	58.9	26.9	0.0089	0.0088	273.5
	D5	32.2	0.0081	0.0081	57.1	26.0	0.0099	0.0098	270.6
(0.5,0.25)	single	45.9	0.0075	0.0075	74.4	22.6	0.0086	0.0085	274.5
	D1	44.6	0.0077	0.0077	70.2	33.5	0.0089	0.0088	273.7
	D2	37.6	0.0079	0.0078	65.3	33.8	0.0091	0.0090	273.1
	D3	43.1	0.0078	0.0078	66.9	34.5	0.0084	0.0083	275.2
	D4	50.8	0.0076	0.0075	74.6	35.3	0.0088	0.0087	273.9
	D5	52.9	0.0077	0.0077	70.9	49.4	0.0082	0.0081	275.8

Table 4. The selected sampling plans for shifted mean

(ϕ,θ)	plan	n_1	a_1	r_1	n_2	r_2	$\hat{\alpha}$	$\hat{\beta}^*$	loss
(0,0)	single	38		2			0.056	0.903	4.72E-02
	D1	22	0	2	57	2	0.100	0.901	1.12E-03
	D2	24	0	2	24	2	0.068	0.904	3.67E-02
	D3	22	0	2	44	2	0.086	0.899	1.45E-02
	D4	22	0	2	57	2	0.100	0.901	1.12E-03
	D5	22	0	2	44	2	0.086	0.899	1.45E-02
(0.25,0.25)	single	45		2			0.081	0.901	2.07E-02
	D1	29	0	2	39	2	0.100	0.900	4.13E-04
	D2	30	0	2	30	2	0.091	0.901	9.82E-03
	D3	25	0	2	50	2	0.096	0.870	3.46E-02
	D4	29	0	2	39	2	0.100	0.900	4.13E-04
	D5	25	0	2	50	2	0.096	0.870	3.46E-02
(0.5,0.25)	single	54		2			0.121	0.896	2.45E-02
	D1	39	0	2	50	3	0.100	0.900	4.90E-04
	D2	38	0	2	38	2	0.125	0.900	2.56E-02
	D3	55	1	3	110	3	0.089	0.901	1.23E-02
	D4	55	1	3	158	3	0.100	0.901	1.24E-03
	D5	55	1	3	110	3	0.089	0.901	1.23E-02

Table 5. Measure of performances for shifted mean

(ϕ,θ)	plan	at AQL				at LTPD			
		ASN cut	AOQ no replace	AOQ replace	ATI	ASN cut	AOQ no replace	AOQ replace	ATI
(0,0)	single	37.3	0.0083	0.0083	52.7	18.9	0.0085	0.0085	274.7
	D1	28.7	0.0082	0.0082	55.5	23.1	0.0093	0.0092	272.5
	D2	27.8	0.0086	0.0085	46.3	24.9	0.0088	0.0087	274.0
	D3	27.5	0.0084	0.0084	50.9	23.1	0.0094	0.0093	272.1
	D4	28.7	0.0082	0.0082	55.5	23.1	0.0093	0.0092	272.5
	D5	27.5	0.0084	0.0084	50.9	23.1	0.0094	0.0093	272.1
(0.25,0.25)	single	43.6	0.0078	0.0078	65.6	21.2	0.0084	0.0084	274.9
	D1	34.7	0.0079	0.0079	61.4	29.8	0.0090	0.0089	273.2
	D2	34.8	0.0080	0.0080	59.1	30.8	0.0089	0.0088	273.7
	D3	31.1	0.0081	0.0080	56.7	26.2	0.0120	0.0119	264.4
	D4	34.7	0.0079	0.0079	61.4	29.8	0.0090	0.0089	273.2
	D5	31.1	0.0081	0.0080	56.7	26.2	0.0120	0.0119	264.4
(0.5,0.25)	single	51.2	0.0072	0.0072	83.8	24.2	0.0085	0.0085	274.5
	D1	47.9	0.0075	0.0075	73.6	40.5	0.0086	0.0084	274.6
	D2	43.7	0.0075	0.0074	76.1	38.8	0.0087	0.0086	274.1
	D3	59.9	0.0073	0.0073	80.6	55.4	0.0082	0.0081	275.8
	D4	61.1	0.0072	0.0072	83.3	55.4	0.0082	0.0081	275.8
	D5	59.9	0.0073	0.0073	80.6	55.4	0.0082	0.0081	275.8

double sampling plans, but larger at LTPD. The more dependence makes the more improvement of AOQ, but replacing the defective items does not improve AOQ much. Since there is no much difference of AOQ at both AQL and LTPD between the plans, acceptable choice might be a plan having smaller ASN. The ATI becomes larger when dependence increases at AQL but not much at LTPD.

7. Conclusion

In this paper, an efficient design procedure of the double sampling plans for the dependent production process models is discussed. In actual implementation of sampling plans, it has been a common occurrence that people just ignore the dependence. But this is due to the lack of suitable methodologies handling the dependent problems not due to the unimportance of dependence. Some QC engineers just took samples far enough apart to avoid dependence. But this effort is not always possible especially when the samples are inspected in order. Although control chart can be used to check the process status, its application is based on the independent observation. Moreover, if producers and consumers risk should be considered when dependence exists, it must not be used.

The specific application requires statistical model identification which seems to be an important further study area. This can be done by observing and collecting the process data using commercial statistical packages. We can assume many dependent process models such as Markov model, Polya process model or time series model, etc. The correct setup of dependent model is important and must be validated. Simulation of the real situation may help in setting up the dependent model. These extra effort can be compensated by avoiding the risk of wrong decision making in sampling plans for dependent production processes.

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96년 8월 최초 접수, 97년 6월 최종 수정

Appendix

Proof of Lemma 1 :

$E[\hat{\gamma}(n_1, a_1, r_1, n_2, r_2)] = E[\overline{S_{n_1, r_1}}] + \sum_{k=a_1+1}^{r_1-1} \{E[\overline{S_{n_1, k}}\ \overline{S_{n_2, r_2-k}}] - E[\overline{S_{n_1, k+1}}\ \overline{S_{n_2, r_2-k}}]\}$

Let us consider the second expectation term of the right hand side first.

$E[\overline{S_{n_1, k}}\ \overline{S_{n_2, r_2-k}}] = Cov[\overline{S_{n_1, k}}, \overline{S_{n_2, r_2-k}}] + E[\overline{S_{n_1, k}}]\ E[\overline{S_{n_2, r_2-k}}]$

The covariance term is

$Cov[\overline{S_{n_1, k}}\ \overline{S_{n_2, r_2-k}}] = Corr[\overline{S_{n_1, k}}, \overline{S_{n_2, r_2-k}}]\sqrt{Var[\overline{S_{n_1, k}}]Var[\overline{S_{n_2, r_2-k}}]}$

$= Corr[\overline{S_{n_1, k}}, \overline{S_{n_2, r_2-k}}]\sqrt{\gamma_{n_1, k}(1-\gamma_{n_1, k})\gamma_{n_2, k}(1-\gamma_{n_2, k})}/m \le 1/(4m).$

Usually replication number m is fairly large, hence $1/(4m) \fallingdotseq 0$. Thus,

$E[\overline{S_{n_1, k}}\ \overline{S_{n_2, r_2-k}}] \fallingdotseq E[\overline{S_{n_1, k}}]\ E[\overline{S_{n_2, r_2-k}}] = \gamma_{n_1, k}\gamma_{n_2, r_2-k}.$

Therefore, the second term becomes

$\sum_{k=a_1+1}^{r_1-1} E[\overline{S_{n_1, k}}\ \overline{S_{n_2, r_2-k}}] \fallingdotseq \sum_{k=a_1+1}^{r_1-1} \gamma_{n_1, k}\gamma_{n_2, r_2-k}.$

Similarly, we can show that $E[\overline{S_{n_1, k+1}}\ \overline{S_{n_2, r_2-k}}] \fallingdotseq \gamma_{n_1, k+1}\gamma_{n_2, r_2-k}.$

Therefore, $E[\hat{\gamma}(n_1, a_1, r_1, n_2, r_2)] \fallingdotseq \gamma_{n_1, r_1} + \sum_{k=a_1+1}^{r_1-1} \{\gamma_{n_1, k}\gamma_{n_2, r_2-k} - \gamma_{n_1, k+1}\gamma_{n_2, r_2-k}\}$

$= \gamma_{n_1, r_1} + \sum_{k=a_1+1}^{r_1-1} \gamma_{n_2, r_2-k}(\gamma_{n_1, k} - \gamma_{n_1, k+1}) = \gamma(n_1, a_1, r_1, n_2, r_2).$

Proof of Proposition 1 :

(1) Since r_2 is the most inner loop variable, variables n_1, a_1, r_1, n_2 are already fixed. Then, we can decide the minimum and maximum values of $\hat{\alpha}$ and $\hat{\beta}^*$ in advance before the actual execution for all r_2 by the monotonous properties. Since $r_1 \le r_2 \le r_1-1+n_2$, $\hat{\alpha}$ has minimum value at $r_2=r_1-1+n_2$ by the row monotonous property. The maximum of $\hat{\alpha}$ occurs at $r_2=r_1$ by the column monotonous property. The proof follows by assigning r_2 to $\hat{\gamma}(n_1, a_1, r_1, n_2, r_2;\text{AQL})$. Thus

$\min\hat{\alpha}(n_1, a_1, r_1, n_2) = \hat{\gamma}(n_1, a_1, r_1, n_2, r_1-1+n_2;\text{AQL})$, and

$\max\hat{\alpha}(n_1, a_1, r_1, n_2) = \hat{\gamma}(n_1, a_1, r_1, n_2, r_1;\text{AQL})$.

(2) Before the execution of n_2 loop, three variables n_1, a_1, r_1 are fixed. Therefore, the interval range must be wider than that of r_2 because n_2 and r_2 are not fixed yet. This implies that the minimum of n_2 loop must be smaller than the minimum of r_2 loop and the maximum of n_2 loop must be larger than the maximum of r_2 loop, i.e.,

$\min\hat{\alpha}(n_1, a_1, r_1) \le \min\hat{\alpha}(n_1, a_1, r_1, n_2)$, and

$\max\hat{\alpha}(n_1, a_1, r_1) \ge \max\hat{\alpha}(n_1, a_1, r_1, n_2)$.

Since $n_1 \le n_2 \le N-n_1$, the minimum occurs when $\min\hat{\alpha}(n_1, a_1, r_1, n_2)$ has $n_2=n_1$ and the maximum occurs when $\max\hat{\alpha}(n_1, a_1, r_1, n_2)$ has $n_2=N-n_1$ by the monotonous property. Thus,

$\min \hat{\alpha}(n_1, a_1, r_1) = \min \hat{\alpha}(n_1, a_1, r_1, n_1) = \hat{\gamma}(n_1, a_1, r_1, n_1, r_1\text{-}1+n_1\ ;\text{AQL})$, and

$\max \hat{\alpha}(n_1, a_1, r_1) = \max \hat{\alpha}(n_1, a_1, r_1, N\text{-}n_1) = \hat{\gamma}(n_1, a_1, r_1, N\text{-}n_1, r_1\ ;\text{AQL})$.

(3) Before the execution of a_1 loop, two variables n_1 and r_1 are fixed. Clearly,

$\min \hat{\alpha}(n_1, a_1) \leq \min \hat{\alpha}(n_1, a_1, r_1)$, and $\max \hat{\alpha}(n_1, a_1) \geq \max \hat{\alpha}(n_1, a_1, r_1)$.

Note that $0 \leq a_1 \leq r_1\text{-}2$. By the acceptance number monotonous property, the minimum occurs when $\min \hat{\alpha}(n_1, a_1, r_1)$ has $a_1 = r_1\text{-}2$, and the maximum occurs when $\max \hat{\alpha}(n_1, a_1, r_1)$ has $a_1 = 0$. Therefore,

$\min \hat{\alpha}(n_1, a_1) = \min \hat{\alpha}(n_1, r_1\text{-}2, r_1) = \hat{\gamma}(n_1, r_1\text{-}2, r_1, n_1, r_1\text{-}1+n_1\ ;\text{AQL})$, and

$\max \hat{\alpha}(n_1, a_1) = \max \hat{\alpha}(n_1, 0, r_1) = \hat{\gamma}(n_1, 0, r_1, N\text{-}n_1, r_1\ ;\text{AQL})$.

(4) At this level only n_1 is fixed. Therefore, $\min \hat{\alpha}(n_1) \leq \min \hat{\alpha}(n_1, a_1)$, and $\max \hat{\alpha}(n_1) \geq \max \hat{\alpha}(n_1, a_1)$. Note that $\min \hat{\alpha}(n_1, a_1)$ and $\max \hat{\alpha}(n_1, a_1)$ are functions of n_1 and r_1. The value $\min \hat{\alpha}(n_1)$ is found from $\min \hat{\alpha}(n_1, a_1)$ as follow. Since n_1 is fixed, the terms $(1\text{-}\overline{S_{n_1, r_1}})$ and $\overline{S_{n_1, r_1}}$ are positive constants. Both $\overline{S_{n_1, r_1}}$ and $\min \hat{\alpha}(n_1, a_1) = \overline{S_{n_1, r_1}} + (\overline{S_{n_1, r_1-1}} - \overline{S_{n_1, r_1}})\overline{S_{n_1, r_1}} = \overline{S_{n_1, r_1}}(1\text{-}\overline{S_{n_1, r_1}}) + \overline{S_{n_1, r_1-1}}\ \overline{S_{n_1, r_1}}$, are non-increasing monotone functions. Therefore, the minimum occurs when $\min \hat{\alpha}(n_1, a_1)$ has $r_1 = n_1$ since $2 \leq r_1 \leq n_1$. Thus, $\min \hat{\alpha}(n_1) = \hat{\gamma}(n_1, n_1\text{-}2, n_1, n_1, 2n_1\text{-}1\ ;\text{AQL})$.

To find $\max \hat{\alpha}(n_1)$, rewrite $\max \hat{\alpha}(n_1, a_1)$ as follow.

$$\max \hat{\alpha}(n_1, a_1) = \overline{S_{n_1, r_1}} + \sum_{k=1}^{r_1-1}(\overline{S_{n_1, k}} - \overline{S_{n_1, k+1}})\overline{S_{N-n_1, r_1-k}} \leq \overline{S_{n_1, r_1}} + \sum_{k=1}^{r_1-1}(\overline{S_{n_1, k}} - \overline{S_{n_1, k+1}})\overline{S_{N-n_1, 1}}$$
$$= \overline{S_{n_1, r_1}} + (\overline{S_{n_1, 1}} - \overline{S_{n_1, r_1}})\overline{S_{N-n_1, 1}} = \overline{S_{n_1, r_1}}(1\text{-}\overline{S_{N-n_1, 1}}) + \overline{S_{n_1, 1}}\ \overline{S_{N-n_1, 1}}.$$

The last equation is a function of only $\overline{S_{n_1, r_1}}$ since $\overline{S_{N-n_1, 1}}$ is a constant. Therefore, the maximum occurs when $\max \hat{\alpha}(n_1, a_1)$ has $r_1 = 2$. Thus, $\max \hat{\alpha}(n_1) = \hat{\gamma}(n_1, 0, 2, N\text{-}n_1, 2\ ;\text{AQL})$

$$= \overline{S_{n_1, 2}} + \sum_{k=1}^{1}(\overline{S_{n_1, k}} - \overline{S_{n_1, k+1}})\overline{S_{N-n_1, 2-k}} = \overline{S_{n_1, 2}} + (\overline{S_{n_1, 1}} - \overline{S_{n_1, 2}})\overline{S_{N-n_1, 1}}.$$