Aircraft Sortie Model Involving a Single Active Target*

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〈Abstract〉

An economic sortie model involving N identical aircrafts attacking a single active target is developed. Using the concepts of Markovian properties, mathematical expressions for the probability of the various events associated with the sortie are obtained. The obtained results are used to derive cost related measures of effectiveness. Then, the most economic sortie time attacking the given target is computed based on maximization of the expected gain of the sortie. A numerical example is provided.

1. Introduction

An economic sortie model involving N identical aircrafts attacking a single active target is developed. Based on realistic assumptions, mathematical expressions for the probabilities of the various events associated with the sortie are obtained by using the concepts of Markovian properties. The obtained results are used to derive cost related measures of effectiveness for the sortie. Then, the most economic sortie time is determined based on maximization of the expected gain of the sortie. A numerical example is provided.

Sivazlian[8] studied an aircraft sortie model related to a single passive target. Recently, Sivazlian and Rhee[9] also developed an aircraft sortie model involving an arbitrary number of passive targets. This research could be considered as modified works of these two sortie models. For more related works in the area of target acquisition and aircraft attack, the reader is referred to Armitage[1], Bailey[2], Builder[4], Fawett and Jones[5], Nyland[6], Quade[7], Snow[10] and so on.

2. Objective

The objective of this research is to derive measures of performance for a sortie through derivation of the probabilities for the various events associated with the sortie based on the concepts of a continuous parameter Markov chain, and then to determine the most economic sortie time. Given that initially N identical aircrafts are involved in the sortie against a single active target, the probability expressions to be computed are $P(1,n,t)$, the probability that at time t, $N-n$ aircrafts have been killed and the target is still alive, and $P(0,n,t)$, the probability that at time t, $N-n$ aircrafts have been killed and the target is also killed, where $n = 0, 1, 2, 3, ..., N$. Once these probabilities are obtained, they will be used to evaluate several measures of performance such as probability of sortie success, probability of sortie failure, expected number of targets killed, expected gain of a sortie and so on. Finally, these results for measures of performance will be associated with appropriate cost elements to determine the most economic sortie time by maximizing the expected gain of the sortie.

* This research was supported financially by Hannam University in 1996.

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3. Problem Statement

Consider a sortie involving N identical aircrafts against a single active target. It is assumed that all aircrafts are 100% combat ready, and that the time origin of the sortie is the time when all aircrafts reach enemy territory simultaneously.

Once the aircrafts reach enemy territory, it is assumed that the time T_i for the i-th aircraft to acquire the target before attacking it, has a negative exponential distribution with parameter μ. Here, it is also assumed that all T_i's are identically and mutually independently distributed. Thus, $1/\mu$ is the average time for each aircraft to acquire a target and $\mu dt + o(dt)$ is the probability that a target is acquired by an aircraft in the time interval $(t, t+dt)$ where $o(dt)/dt$ is zero as tending $dt \to 0$. Once acquired, it is assumed that the target is attacked by the aircraft and that the attack time is negligible. The probability of the target killed once attacked is assumed to be p_2. More than one attacking on the target during an infinitesimal time interval is a negligible event. If the active target is once attacked and not killed, it is assumed that it moves to an unidentified location immediately to hide. This situation leads to all N aircrafts go into the process of reacquiring the hidden target again, similar to the beginning of the sortie. Hence, without loss of generality, it is also assumed that hidden target reacquisition time of an aircraft is a random variable which is identically distributed to T_i, and the same procedures repeat until the sortie is completed.

From the moment the aircrafts enter enemy territory, they are surrounded by a hostile environment which takes the form of enemy threat. It is assumed that the occurrence of the enemy threats is a Poisson process with intensity λ. This asserts that the average number of enemy threats per unit time encountered for each aircraft is λ. Equivalently, $\lambda dt + o(dt)$ is the probability that an aircraft will encounter an enemy threat in the time interval $(t, t+dt)$. Once an enemy threat is encountered, the probability that the aircraft is killed is p_1.

Finally, the aircrafts is assumed to carry an adequate supply of weapons and ammunitions to possibly destroy the target.

It should be stated that the assumptions made above for simplicity of the model in this research, for example, the N identical aircrafts, the probabilities of the targe killed and an aircraft killed p_2 and p_1, respectively, and so on, may not encompass all real situations. In particular, even though p_2 and p_1 could be obtained from various known approaches like Gaussian or uniform kill function, such procedures are omitted. However, the model developed in this research could still provide basic information as a prototype model to develop alternative or more realistic ones.

4. Analysis of the Model

Let T be the random variable representing the target acquisition time when $(N-n)$ aircrafts are alive. Since it is assumed that target acquisition time for each aircraft is T_i which is exponentially distributed with parameter μ, T could be expressed by

$$T = \min[T_1, T_2, \ldots, T_{N-n}] \tag{1}$$

Hence, based on (1), the probability density function of T, denoted by $f_T(t)$, is given by

$$f_T(t) = (N-n)\mu e^{-(N-n)\mu t}, \quad t \geq 0.$$

Note that minimum of mutually independent exponential distributions is also an exponential distribution. In order to analyze the problem under consideration, recall $P(i,n,t)$ where i=0,1 and n = 0,1,2,...,N. Note that i=0 corresponds to the state 'target is killed', and i=1 corresponds to the state 'target is not killed'. Also note that n denotes $N-n$ aircrafts have been killed. Based on the Markovian properties, the following a set of difference equations could be constructed (see, e.g., Bhat [3]):

$$P(1,N,t+dt) = P(1,N,t)(1-Np_1\lambda dt-Np_2\mu dt) + o(dt) \tag{2}$$

$$\begin{aligned} P_N(1,N-n,t+dt) &= P_N(1,N-n,t)[1-(N-n)p_1\lambda dt][1-(N-n)p_2\mu dt] \\ &\quad + P(1-N-n+1,t)[1-(N-n+1)p_2\mu dt][(N-n+1)p_1\lambda dt] + o(dt), \end{aligned}$$

for n = 1, 2, 3, ..., N,

$$P(0,N,t+dt) = P(0,N,t)[1-Np_1\lambda dt] + P(1,N,t)Np_2\mu dt + o(dt)$$

$$P[0,N-n,t) = P(0,N-n,t)[1-(N-n)p_1\lambda dt] + P(0,N-n+1,t)[(N-n+1)p_1\lambda dt] + P(1,N-n,t)[(N-n)p_2\mu dt] + o(dt),$$

for $n = 1, 2, 3, ..., N$.

Using the above difference equations, after rearranging terms, divide by dt and tend $dt \rightarrow 0$ yield a set of differential equations. From relation (2), for example, the following can be obtained:

$$\frac{d}{dt}P(1,N,t) + N(p_1\lambda + p_2\mu)P(1,N,t) = 0. \quad (3)$$

Then solving such difference equations in, for example, (3) with the initial condition such that

$$P(1,N,0) = 1$$

yields $P(i,n,t)$ explicitly for all i and n. For example,

$$P(1,N-n,t) = [\frac{\alpha}{\alpha+\beta}]^n \begin{pmatrix} N \\ m \end{pmatrix} [1-e^{-(\alpha+\beta)t}]^n [e^{-(\alpha+\beta)t}]^{N-n},$$

$$n = 0, 1, 2, ..., N$$

where $\alpha = p_1\lambda$, $\beta = p_2\mu$ and $\begin{pmatrix} N \\ m \end{pmatrix} = \frac{N!}{n!(N-n)!}$.

Since the procedures to obtain other $P(i,n,t)$'s are straight forward, the final forms for other cases are omitted in this research and left to the readers. Note that setting $N=1$ in the obtained $P(i,n,t)$ above, yields the results obtained by Sivazlian [8].

5. Measures of Sortie Performance

The following measures of performance commonly used in the area of the aircraft sortie could be considered:

(i) the probability of sortie success,
(ii) the probability of sortie failure,
(iii) the expected number of targets killed,
(iv) the expected number of attacks on the target,
(v) the expected number of aircrafts killed,
(vi) the expected duration of sortie.

(1) The probability of sortie success

This may be defined in different ways:

(i) The probability that the target is killed and none of the aircrafts are killed, i.e., $P(0,N,t)$.
(ii) The probability that the target is killed, i.e., $\Sigma_{n=0}^{N} [P(0,N-n,t)+P(1,N-n,t)]$

(2) The probability of sortie failure

This could also be express in different ways:

(i) The probability that the target is not killed and all aircrafts are killed, i.e., $P(1,0,t)$
(ii) The probability that the target is not killed and at least one aircraft is killed, i.e., $\Sigma_{n=0}^{N-1} P(1,N-n,t)$
(iii) The probability that all aircrafts are killed, i.e., $P(1,0,t) + P(0,0,t)$

(3) The expected number of targets killed

Let $E[Q(t)]$ be the expected number of targets killed at time t. Then, $E[Q(t)]$ is obtained from

$$E[Q(t)] = \Sigma_{n=0}^{N} P_N(0,N-n,t) \quad (4)$$

(4) The expected number of attacks on the target

Let $A(t)$ be the random variable denoting the number of attacks on the target by an aircraft at time t. Since an attacking is made only when the target is not killed, define $P(m,i,j,t)$ as the probability that at time t the number of attacks on the target is m ($m = 0, 1, 2, 3, ...$) and the target is in state i ($i = 0, 1$) and the aircraft in state j ($j = 0, 1$), where $i = 0$ ($j = 0$) corresponds the target (aircraft) is killed and $i = 1$ ($j = 1$) corresponds the target (aircraft) is not killed. Then, based on Markovian properties, the following difference equations hold:

$$P(0,1,1,t+dt) = P(0,1,1,t)(1-\alpha dt-\mu dt) + o(dt)$$

$$P(m,1,1,t+dt) = P(m,1,1,t)(1-\alpha dt-\mu dt) + P(m-1,1,1,t)(\mu-\beta)dt + o(dt)$$
for $m = 1, 2, 3, \ldots$

With the following initial condition

$$P(m,i,j,0) = 1, \text{ if } m=0,\ i=1 \text{ and } j=1.$$

$P(m,1,1,t)$ is obtained as

$$P(m,1,1,t) = \frac{(\mu t-\beta t)^n}{m!} e^{-(\alpha+\mu)t} \quad \text{for } m = 0, 1, 2, 3, \ldots$$

Similar approaches yield $P(m,0,1,t)$, $P(m,1,0,t)$, $P(m,0,0,t)$ explicitly. Thus, the expected number of attacks on the target by an aircraft, denoted by $E[A(t)]$ could be obtained from

$$\text{l}(t)] = \Sigma_{m=0}^{\infty} m[P(m,1,1,t) + P(m,0,1,t) + P(m,1,0,t) + P(m,0,0,t)]$$

Since an N aircrafts are involved in the sortie, the total expected number of attacks on the target up to time t, denoted by $E[TA(t)]$, is given by

$$E[TA(t)] = NE[A(t)] \quad (5)$$

(5) The expected number of aircrafts killed

Let $E[N(t)]$ be the expected number of aircrafts killed up to time t. Then $E[N(t)]$ is obtained from the following relation:

$$E[N(t)] = \Sigma_{n=0}^{N} n[P(1,N-n,t) + P(0,N-n,t)] \quad (6)$$

(6) The expected duration of the sortie

The expected duration of the sortie is limited by the expected waiting time until all aircrafts are killed. Since the expected waiting time until the last aircraft is killed, is equal to the maximal expected waiting time among the N aircrafts (assuming no limitations on fuel tank capacity). For every aircraft, the expected waiting time until it is killed is $1/\alpha$. In other words, the probability that one aircraft is killed between time interval x and $x+dx$ is $\alpha e^{-\alpha x}dx$. Thus, the probability density function of duration of sortie, denoted by $f_D(x)$, can be obtained based on the following relation:

$$D = \max[W_1, W_2, \ldots, W_N]$$

Note that the W_i's are identically, independently and exponentially distributed with parameter α. Therefore,

$$f_D(x) = N\alpha e^{-\alpha x}(1-e^{-\alpha x})^{N-1}, \quad x > 0$$

If the length of the sortie is t, the duration of sortie, denoted by $D(t)$ is

$$D(t) = \begin{cases} x, & 0 \le x < t \\ t, & t \le x < \infty \end{cases}$$

Finally, the expected duration of the sortie, denoted by $E[D(t)]$, is obtained from

$$E[D(t)] = \int_0^t x f_D(x)dx + \int_t^{\infty} t f_D(x)dx \quad (7)$$

6. Economic Analysis of the Sortie

In practical situations, cost play an important role in decision making process. To be meaningful, define the following cost elements:

(i) fixed cost in preparation of the sortie, K
(ii) cost per unit duration time for each aircraft, C_D,
(iii) replacement cost of each aircraft killed and cost of crew loss, C_A
(iv) cost of ammunition per attack on a target, C_M
(v) monetary benefit in killing a single active target, B.

Also let $E[G(t)]$ be the expected gain of the sortie. Then, measurements of the sortie performance and their corresponding cost elements defined above, together yields,

$$E[G(t)] = BE[Q(t)] - (K + NC_D E[D(t)] + C_A E[N(t)] + C_M E[TA(t)]) \quad (8)$$

Note that $E[Q(t)]$, $E[D(t)]$, $E[N(t)]$ and $E[TA(t)]$ are given in (4), (7), (6) and (5), respectively. Clearly, $E[G(t)]$ is a function t. Hence, maximizing $E[G(t)]$ in (8) could determine the optimal sortie time. However, it is very difficult to determine the optimal value of decision variable, t^* analytically. Hence, numerical approaches are suggested.

7. Numerical example.

The following values of input parameters and cost elements are assumed [9]:

(i) $\lambda = 7/hr$, $\mu = 4/hr$, $p_1 = 0.2$, $p_2 = 0.1$

(ii) K = \$2,000/aircraft, C_D = \$6,000/aircraft for an hour of the sortie, C_A = \$30,000,000/aircraft and crew, C_M = \$5,000/attack, B = \$150,000,000.

The optimal sortie times when the number of aircrafts are given, are shown in Table 1. For example, when 5 aircrafts are dispatched, the optimal sortie time is 12 minutes.

If all values of parameters above are fixed except λ, the optimal sortie times for different values of λ are shown in Table 2. If all values of parameters above are fixed except p_1, the optimal sortie times for different values of p_1 are shown in ⟨Table 3⟩.

8. Summary

In this research, a sortie model involving N identical aircrafts against single active target is considered. Based on the concepts of Markovian properties, various probabilities associated with the sortie are derived. These probabilities is used to obtained measures of sortie performance. With necessary cost elements, the expected gain of the sortie is able to obtain. The most economic sortie time by maximizing the expected gain of the sortie could be determined. Even though assumptions made may not be completely appropriate in real situations, this model could provide fundamental insights of the sortie problem involving active targets. Hence, a number of other sortie models could be developed by relaxing the restrictions imposed to the model such as different types of aircrafts involved in a sortie, attack on multiple active targets and do on.

⟨Table 1⟩ The most economic sortie time

N	t^*(min)	$E[G(t^*)]$ (\$)
2	32	6,734,792
3	21	7,166,612
4	15	7,325,577
5	12	7,399,418
6	10	7,436,281
7	8	7,455,689
8	7	7,473,509
9	6	7,487,051
10	6	7,494,209

⟨Table 2⟩ Variation of λ

λ	N	t^*(min)	$E[G(t^*)]$ (\$)
6.0	30	3	13,751,265
7.0	28	2	7,516,352
8.0	26	1	3,043.256

⟨Table 3⟩ Variation of p_2

p_1	N	t^*(min)	$E[G(t^*)]$ (\$)
0.15	30	3	20,314,840
0.18	28	3	11,491,171
0.20	28	2	7,516,352
0.22	28	1	3,945,752
0.25	20	1	1,212,606

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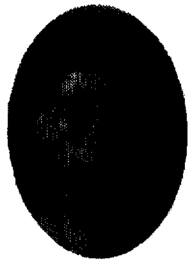

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