An Alternative Complex Variable Method in Plane Elasticity

BEONG IN YUN

Department of Computer Science and Statistics,
Kunsan National University, 573-701 Kunsan, Korea

Abstract

For two dimensional elasticity, we suggest a new complex variable method using the Navier's displacement equation. This method gives alternative displacement and stress formulae to those resulting from the Muskhelishvili's complex function method.

1 Introduction

Functions of a complex variable were introduced into plane elastic problems in 1909 by Kolosov. The resulting developments have been described by Muskhelishvili [1], Sokolnikoff [2], England [3], etc. There are two well-known fundamental methods, say, Westergaard method and Muskhelishvili's complex potential method, both of which are based on the Airy stress functions.

The Westergaard method constitutes a simple versatile tool for solving a certain class of plane elasticity problems. However, in general, this method is not available for the domain of which boundary is not simple nor for various boundary conditions. Muskhelishvili generalized the Airy stress function for the two-dimensional elasticity by using two analytic functions, $\phi(z)$ and $\chi(z)$, say, complex potentials. The function ϕ is related to the Airy stress function, and χ is an arbitrary analytic function, which are defined in the next section.

It is known that the plane elasticity problems reduce to the solution of Navier's displacement equations subjected to certain boundary conditions. In this paper, using these Navier equations, we present an alternative complex variable method to the Muskhelishvili's complex function method. On writing the Navier equations in complex variable notation two real valued functions, $h(x_1, x_2)$ and $g(x_1, x_2)$ are introduced, which consist of the first derivatives of the displacement components. This results in an analytic function composed of the functions, $h(x_1, x_2)$ and $g(x_1, x_2)$. Using the Hooke's law and strain - displacement relationship, we obtain the equivalent formulae of the stresses and displacements to those derived by the Muskhelishvili's method.

2 Muskhelishvili's Complex Function Method

In this section, to compare the present method with those developed before, we summarize the Muskhelishvili's complex function method [1].

2.1. *Determination of the Displacements from the Airy Stress Function*

Let the region Ω in the plane be simply connected and let ψ be the Airy stress function such as

$$\sigma_{xx} = \frac{\partial^2\psi}{\partial y^2}, \qquad \sigma_{yy} = \frac{\partial^2\psi}{\partial x^2}, \qquad \sigma_{xy} = \frac{\partial^2\psi}{\partial x \partial y}\,. \tag{1}$$

Equilibrium equation of the stresses implies that the Airy stress function satisfies the biharmonic equation, that is,

$$\Delta^2\psi = 0\,.$$

To find the displacements u_1 and u_2, using the Airy stress function ψ, we consider the following equations which may be derived from the equations in (1), the Hooke's law with the plane strain condition and the strain - displacement relation :

$$\begin{aligned} &\lambda\theta + 2\mu\frac{\partial u_1}{\partial x} = \frac{\partial^2\psi}{\partial y^2}\,, \qquad \lambda\theta + 2\mu\frac{\partial u_2}{\partial y} = \frac{\partial^2\psi}{\partial x^2}\,, \\ &\mu\left(\frac{\partial u_2}{\partial x} + \frac{\partial u_1}{\partial y}\right) = -\frac{\partial^2\psi}{\partial x \partial y}\,, \end{aligned} \tag{2}$$

where $\theta = \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y}$. From the first two of these equations,

$$2(\lambda+\mu)\theta = \Delta\psi \qquad \text{i.e.} \quad \lambda\theta = \frac{\lambda}{2(\lambda+\mu)}\,\Delta\psi\,,$$

so that we have

$$2\mu\frac{\partial u_1}{\partial x} = \frac{\partial^2\psi}{\partial y^2} - \frac{\lambda}{2(\lambda+\mu)}\,\Delta\psi\,, \qquad 2\mu\frac{\partial u_2}{\partial y} = \frac{\partial^2\psi}{\partial x^2} - \frac{\lambda}{2(\lambda+\mu)}\,\Delta\psi\,.$$

Introducing a function $P(x, y)$ defined as

$$\Delta\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = P\,, \tag{3}$$

it follows that

$$2\mu\frac{\partial u_1}{\partial x} = -\frac{\partial^2\psi}{\partial x^2} + \frac{\lambda+2\mu}{2(\lambda+\mu)}\,P\,, \qquad 2\mu\frac{\partial u_2}{\partial y} = -\frac{\partial^2\psi}{\partial y^2} + \frac{\lambda+2\mu}{2(\lambda+\mu)}\,P \tag{4}$$

From (3), we note that

$$\Delta P = \Delta^2\psi = 0\,,$$

that is, $P(x, y)$ is a *harmonic* function.

Now let $Q(x, y)$ be a harmonic conjugate function of $P(x, y)$ such that

$$f(z) = P(x, y) + i\,Q(x, y) \tag{5}$$

is an analytic function. The function Q may be determined for a given P apart from an arbitrary constant. Furthermore, if we take a function ϕ as

$$\phi(z) = p + iq = \frac{1}{4}\int f(z)\,dz\,, \tag{6}$$

then, since ϕ is analytic,

$$\phi'(z) = \frac{\partial p}{\partial x} + i\,\frac{\partial q}{\partial x} = \frac{1}{4}(P + i\,Q)\,.$$

Thus

$$\frac{\partial p}{\partial x} = \frac{\partial q}{\partial y} = \frac{1}{4}P\,, \qquad \frac{\partial p}{\partial y} = -\frac{\partial q}{\partial x} = -\frac{1}{4}Q\,. \tag{7}$$

Substituting $P = 4\frac{\partial p}{\partial x} = 4\frac{\partial q}{\partial y}$ into the formulae (4), we have

$$2\mu\frac{\partial u_1}{\partial x} = -\frac{\partial^2\psi}{\partial x^2} + \frac{2(\lambda+2\mu)}{\lambda+\mu}\,\frac{\partial p}{\partial x}\,, \qquad 2\mu\frac{\partial u_2}{\partial y} = -\frac{\partial^2\psi}{\partial y^2} + \frac{2(\lambda+2\mu)}{\lambda+\mu}\,\frac{\partial q}{\partial y}\,.$$

Integration gives

$$2\mu u_1 = -\frac{\partial\psi}{\partial x} + \frac{2(\lambda+2\mu)}{\lambda+\mu}\,p + f_1(y)\,, \qquad 2\mu u_2 = -\frac{\partial\psi}{\partial y} + \frac{2(\lambda+2\mu)}{\lambda+\mu}\,q + f_2(x)\,.$$

Substituting these expressions into the third equation in (2) and noting that

$$\frac{\partial p}{\partial y} + \frac{\partial q}{\partial x} = 0\,,$$

we have

$$f_1'(y) + f_2'(x) = 0\,,$$

and thus

$$f_1(y) = c_3 y + c_1, \qquad f_2(x) = -c_3 x + c_2\,,$$

where $c_1,\ c_2$ and c_3 are arbitrary constants. In this equations, f_1 and f_2 mean the rigid body motion for the displacements.

Omitting the rigid body displacements, the following displacement formulae are attained :

$$2\mu u_1 = -\frac{\partial \psi}{\partial x} + \frac{2(\lambda+2\mu)}{\lambda+\mu}\,p\,, \qquad 2\mu u_2 = -\frac{\partial \psi}{\partial y} + \frac{2(\lambda+2\mu)}{\lambda+\mu}\,q\,. \tag{8}$$

2.2. *Complex Representation of the Displacements and Stresses*

Noting that $\Delta\psi = P$ and that, from (7),

$$\begin{aligned}\Delta(px+qy) &= x\,\Delta p + y\,\Delta q + 2\left(\frac{\partial p}{\partial x} + \frac{\partial q}{\partial y}\right)\\ &= 4\frac{\partial p}{\partial x} = P\,,\end{aligned}$$

it follows that

$$\Delta(\psi - px - qy) = 0\,.$$

Thus the Airy stress function may be written as

$$\psi(x,y) = px + qy + p_1\,,$$

where p_1 is some harmonic function. Now, let

$$\chi(z) = p_1 + i q_1\,,$$

where q_1 is a harmonic conjugate to p_1. Since the region is assumed to be simply connected, χ is analytic.

The Airy stress function ψ can be rewritten by

$$\psi(x,y) = \mathrm{Re}\left[\bar{z}\phi(z) + \overline{\chi(z)}\right]\,, \tag{9}$$

or

$$2\psi(x,y) = \overline{z}\phi(z) + z\,\overline{\phi(z)} + \chi(z) + \overline{\chi(z)}\,. \tag{10}$$

Noting that $\frac{\partial}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \overline{z}}$ and $\frac{\partial}{\partial y} = i\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \overline{z}}\right)$, it is easily found that

$$\begin{aligned} 2\frac{\partial\psi}{\partial x} &= \phi(z) + \overline{z}\,\phi'(z) + \overline{\phi(z)} + z\,\overline{\phi'(z)} + \chi'(z) + \overline{\chi'(z)}\,, \\ 2\frac{\partial\psi}{\partial y} &= i\left\{-\phi(z) + \overline{z}\,\phi'(z) + \overline{\phi(z)} - z\,\overline{\phi'(z)} + \chi'(z) - \overline{\chi'(z)}\right\}\,. \end{aligned} \tag{11}$$

To derive the formulae of the displacements and stresses, it will be more convenient to deal with the expression

$$\frac{\partial\psi}{\partial x} + i\frac{\partial\psi}{\partial y} = \phi(z) + z\,\overline{\phi'(z)} + \overline{\chi'(z)}\,. \tag{12}$$

From the equation (8) with $\phi(z) = p + iq$ and (12), we have

$$\begin{aligned} 2\mu(u_1 + iu_2) &= -\left(\frac{\partial\psi}{\partial x} + i\,\frac{\partial\psi}{\partial y}\right) + \frac{2(\lambda+2\mu)}{\lambda+\mu}\,\phi(z) \\ &= \kappa\phi(z) - z\,\overline{\phi'(z)} - \overline{\chi'(z)}\,, \end{aligned} \tag{13}$$

where

$$\kappa = \frac{\lambda+3\mu}{\lambda+\mu} = 3 - 4\nu\,,$$

for the plane strain condition which we have assumed when the equations (2) are derived. In the case of the plane stress, the formula (13) is available when we replace the Poisson's ratio ν by $\frac{\nu}{1+\nu}$, that is, $\kappa = (3-\nu)/(1+\nu)$. Next, for the representation of the stresses, using (1),

$$\begin{aligned} \sigma_{xx} + i\sigma_{xy} &= \frac{\partial}{\partial y}\left(\frac{\partial\psi}{\partial y} - i\,\frac{\partial\psi}{\partial x}\right) \\ &= -i\,\frac{\partial}{\partial y}\left(\frac{\partial\psi}{\partial x} + i\,\frac{\partial\psi}{\partial y}\right) \\ &= \left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \overline{z}}\right)\left(\frac{\partial\psi}{\partial x} + i\,\frac{\partial\psi}{\partial y}\right)\,. \end{aligned}$$

Substitution of (12) into this equation gives

$$\sigma_{xx} + i\sigma_{xy} = \phi'(z) + \overline{\phi'(z)} - z\,\overline{\phi''(z)} - \overline{\chi''(z)}\,. \tag{14}$$

Similarly, since

$$\begin{aligned}\sigma_{yy} - i\sigma_{xy} &= \frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial x} + i\,\frac{\partial\psi}{\partial y}\right)\\ &= \left(\frac{\partial}{\partial z} + \frac{\partial}{\partial\overline{z}}\right)\left(\frac{\partial\psi}{\partial x} + i\,\frac{\partial\psi}{\partial y}\right),\end{aligned}$$

we have

$$\sigma_{yy} - i\sigma_{xy} = \phi'(z) + \overline{\phi'(z)} + z\,\overline{\phi''(z)} + \overline{\chi''(z)}\,. \tag{15}$$

In addition, equations (14) and (15) result in

$$\sigma_{xx} + \sigma_{yy} = 2\left\{\phi'(z) + \overline{\phi'(z)}\right\} = 4\,\mathrm{Re}[\phi'(z)] \tag{16}$$

$$\sigma_{xx} - \sigma_{yy} + 2i\sigma_{xy} = -2\left\{z\,\overline{\phi''(z)} + \overline{\chi''(z)}\right\}\,. \tag{17}$$

3 Alternative Method Using the Navier Equations

In this section we have developed a new complex variable method for the plane elasticity, which results in the equivalent formulae of the displacements and stresses to the Muskhelishvili's ones.

The displacement vector $\mathbf{u} = u_1 + iu_2$ satisfies the following Navier equation :

$$\mu\Delta\mathbf{u} + (\lambda + \mu)\mathrm{div}(\mathrm{grad}\ \mathbf{u}) = 0\,, \tag{18}$$

or

$$\begin{aligned}\mu\Delta u_1 + (\lambda + \mu)\frac{\partial}{\partial x_1}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\right) &= 0\\ \mu\Delta u_2 + (\lambda + \mu)\frac{\partial}{\partial x_2}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\right) &= 0\,,\end{aligned} \tag{19}$$

where the Lamé constant λ, shear modulus μ and the constant κ are such as

$$\mu = \frac{E}{2(1+\nu)}\,, \quad \lambda = \frac{2\mu\nu}{1-2\nu}\,,$$

and

$$\kappa = \begin{cases} 3 - 4\nu & \text{for plane strain condition} \\ \frac{3-4\nu}{1+\nu} & \text{for plane stress condition}\,. \end{cases}$$

Let

$$\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} = h(x_1, x_2) \qquad \text{and} \quad \frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1} = g(x_1, x_2)\,. \tag{20}$$

Then

$$\Delta u_1 = \frac{\partial h}{\partial x_1} + \frac{\partial g}{\partial x_2} \qquad \text{and} \quad \Delta u_2 = \frac{\partial h}{\partial x_2} - \frac{\partial g}{\partial x_1}\,,$$

so that the equation (19) becomes

$$\begin{aligned} (\lambda + 2\mu)\frac{\partial h}{\partial x_1} + \mu\frac{\partial g}{\partial x_2} &= 0\,, \\ (\lambda + 2\mu)\frac{\partial h}{\partial x_2} - \mu\frac{\partial g}{\partial x_1} &= 0\,. \end{aligned} \tag{21}$$

If we take a complex function

$$f(z) = (\lambda + 2\mu)\, h(x_1, x_2) - i\mu\, g(x_1, x_2)\,, \qquad z = x_1 + ix_2\,, \tag{22}$$

then, by the relations

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right), \qquad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right),$$

equation (21) implies that

$$2\frac{\partial}{\partial \overline{z}} f(z) = 0\,, \quad \text{that is,} \quad f(z) \text{ is analytic}\,.$$

Now, from the definition of $f(z)$ in (22),

$$h(x_1, x_2) = \frac{1}{2(\lambda + 2\mu)}\left\{f(z) + \overline{f(z)}\right\} \quad \text{and} \quad g(x_1, x_2) = \frac{i}{2\mu}\left\{f(z) - \overline{f(z)}\right\}\,. \tag{23}$$

If we take an analytic function $\phi(z)$ and set

$$f(z) = \frac{4\nu}{\lambda}(\lambda + 2\mu) \cdot \phi'(z)\,, \tag{24}$$

for simplicity, then the equations in (23) give

$$
\begin{aligned}
h(x_1,x_2) &= 2\left(\frac{\nu}{\lambda}\right)\left\{\phi'(z)+\overline{\phi'(z)}\right\}\,,\\
g(x_1,x_2) &= 2i\left(\frac{\nu}{\mu}\right)\left(1+2\frac{\mu}{\lambda}\right)\left\{\phi'(z)-\overline{\phi'(z)}\right\}\,.
\end{aligned} \tag{25}
$$

On the other hand, from the relation (20),

$$
2\frac{\partial}{\partial z}(u_1+iu_2) = h(x_1,x_2) - ig(x_1,x_2)\,,
$$

so that

$$
2(u_1+iu_2) = \int h(x_1,x_2) - ig(x_1,x_2)\,dz + \overline{R(z)}\,, \tag{26}
$$

where $R(z)$ is an arbitrary analytic function.
By substituting (25) into (26),

$$
\begin{aligned}
&2(u_1+iu_2)\\
&\quad = \frac{1}{\mu}\int\left(\frac{2\nu\mu}{\lambda}\right)\left\{\phi'(z)+\overline{\phi'(z)}\right\} + 2\nu\left(1+2\frac{\mu}{\lambda}\right)\left\{\phi'(z)-\overline{\phi'(z)}\right\}dz + \overline{R(z)}\\
&\quad = \frac{1}{\mu}\int(1-2\nu)\left\{\phi'(z)+\overline{\phi'(z)}\right\} + 2(1-\nu)\left\{\phi'(z)-\overline{\phi'(z)}\right\}dz + \overline{R(z)}\\
&\quad = \frac{1}{\mu}\left[\kappa\phi(z) - z\overline{\phi'(z)}\right] + \overline{R(z)}\,.
\end{aligned}
$$

For an analytic function $\xi(z)$, if we take $R(z) = -\frac{1}{\mu}\xi(z)$ then we have

$$
2\mu(u_1+iu_2) = \kappa\phi(z) - z\overline{\phi'(z)} - \overline{\xi(z)}\,. \tag{27}
$$

We identify the coordinates as $x_1 = x,\ x_2 = y$ and denote the strain components as

$$
\epsilon_{xx} = \frac{\partial u_1}{\partial x}, \qquad \epsilon_{yy} = \frac{\partial u_2}{\partial y}\,.
$$

Using Hooke's law and (25), for the plane strain condition, we have

$$
\begin{aligned}
\sigma_{xx}+\sigma_{yy} &= \frac{E}{(1+\nu)(1-2\nu)}\{\epsilon_{xx}+\epsilon_{yy}\} = \left(\frac{\lambda}{\nu}\right)\{\epsilon_{xx}+\epsilon_{yy}\}\\
&= \left(\frac{\lambda}{\nu}\right)h(x_1,x_2)\\
&= 2\left\{\phi'(z)+\overline{\phi'(z)}\right\}\,.
\end{aligned} \tag{28}
$$

By the Hooke's law and the strain - displacement relation it follows that

$$
\begin{aligned}
\sigma_{xx} - \sigma_{yy} + 2i\sigma_{xy} &= \frac{E}{1+\nu}\left\{(\epsilon_{xx} - \epsilon_{yy}) + 2i\epsilon_{xy}\right\} \\
&= 2\mu\left\{\left(\frac{\partial u_1}{\partial x_1} - \frac{\partial u_2}{\partial x_2}\right) + i\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)\right\} \\
&= 4\mu\frac{\partial}{\partial \overline{z}}(u_1 + iu_2)\,.
\end{aligned}
$$

Substituting (27) into this equation, we have

$$\sigma_{xx} - \sigma_{yy} + 2i\sigma_{xy} = -2\left\{z\overline{\phi''(z)} + \overline{\xi'(z)}\right\}\,. \tag{29}$$

Subtraction of (29) from (28) gives

$$\sigma_{yy} - i\sigma_{xy} = \phi'(z) + \overline{\phi'(z)} + z\overline{\phi''(z)} + \overline{\xi'(z)}\,. \tag{30}$$

Addition of (29) to (30) gives

$$\sigma_{xx} + i\sigma_{xy} = \phi'(z) + \overline{\phi'(z)} - z\overline{\phi''(z)} - \overline{\xi'(z)}\,. \tag{31}$$

We can observe that, if we set $\xi(z) = \chi'(z)$, the formulae (27) – (31) are equivalent to the Muskhelishvili's formulae in (13) and (17). It should be noted that the present approach is simple in calculation and for explanation compared with the usual methods.

Complex variable representation of the displacements and stresses is useful for the analysis of the plane elasticity, in general. In fact, it has been shown in the literature that the complex variable method is available for many problems in linear elastic fracture mechanics. Particularly, the representation in (27)-(31) or in (13)-(17) provides principal basis on the formulation of the boundary integral equations to solve the various crack problems in the plane. One can expect that proper variations of the formulae in (27)-(31) may result in the efficient numerical schemes for some practical applications such as interface crack problems.

References

[1] N. I.Muskhelishvili(1977), Some basic problems of the mathematical theory of elasticity, 4 th edition, Noordhoff international Publ., Netherlands.

[2] I. S. Sokolnikoff(1956), Mathematical theory of elasticity McGraw - Hill, New York, 1956.

[3] A. H.England(1971), Complex variable methods in elasticity, Wiley - Interscience, U. K., 1971.