Journal of Statistical
Theory & Methods
1997, Vol. 8, No. 2, pp. 135 ~ 141

Note on the Transformed Geometric Poisson Processes[1]

Jeong Hyun Park [2]

Abstract

In this paper, it is investigated the properties of the transformed geometric Poisson process when the intensity function of the process is a distribution of the continuous random variable. If the intensity function of the transformed geometric Poisson process is a Pareto distribution then the transformed geometric Poisson process is a strongly P-process.

Key Words and Phrases: P-process, strongly P-process, transformed geometric Poisson process.

1. Introduction

Park(1997) introduced the P-process and the transformed geometric Poisson process such that the intensity function is $g_i(t) \neq g_j(t)$ for $i \neq j$. In general, the transformed geometric Poisson process is not a strongly P-process. In this paper, we will introduce the transformed geometric Poisson process which is a strongly P-process. Let $\{N(t)|t \geq 0\}$ be a counting process having jump magnitude 1. Then counting process $\{N(t)|t \geq 0\}$ satisfies

$$P\{N(t+h) - N(t) \geq 2\} = o(h). \tag{1}$$

Suppose $P\{N(t+h)-N(t) = 1|N(t) = n\} = g_n(t,h)$ and $g_n(t,h)$ is a polynomial function with respect to h in which the constant term is zero. That is, we can express

$$g_n(t,h) = g_n(t)h + g_n(t)h^2 + \cdots = g_n(t)h + o(h).$$

[1]This paper was supported by research fund, Kwangdong University, 1997
[2]Associate Professor, Department of Computer Science and Statistics, Kwandong University, Kangnung, 210-701

Then

$$P\{N(t+h)-N(t)=1|N(t)=n\}=g_n(t)h+o(h), \tag{2}$$

and

$$P\{N(t+h)-N(t)=0|N(t)=n\}=1-g_n(t)h+o(h).$$

Now $g_n(t)$ is called the *intensity function* of the counting process $\{N(t)|t\geq 0\}$. By equations (1) and (2),

$$\begin{aligned} P\{N(t+h)=n\} &= P\{N(t+h)-N(t)=0|N(t)=n\}P\{N(t)=n\} \\ &\quad +P\{N(t+h)-N(t)=1|N(t)=n-1\}P\{N(t)=n-1\} \\ &\quad +\sum_{i=2}^{\infty} P\{N(t+h)-N(t)=i|N(t)=n-i\}P\{N(t)=n-i\} \\ &= \{1-g_n(t)h\}P\{N(t)=n\}+g_{n-1}(t)hP\{N(t)=n-1\}+o(h). \end{aligned}$$

Hence,

$$\begin{aligned} \frac{P\{N(t+h)=n\}-P\{N(t)=n\}}{h} &= g_n(t)P\{N(t)=n\} \\ &\quad +g_{n-1}(t)P\{N(t)=n-1\}+\frac{o(h)}{h}. \end{aligned}$$

Letting $h \longrightarrow 0$, we obtain differential equations

$$\frac{dP\{N(t)=n\}}{dt}=g_n(t)P\{N(t)=n\}+g_{n-1}(t)P\{N(t)=n-1\}.$$

The solution of above equations are

$$P\{N(t)=n\}=e^{-\int g_n(t)dt}\int g_{n-1}(t)P\{N(t)=n-1\}e^{\int g_n(t)dt}dt+k_n e^{-\int g_n(t)dt}.$$

If the countig process is a Poisson or nonhomogeneous Poisson, we know that $k_0=1$ and $k_n=0(n\geq 1)$. The constants $\{k_0,k_1,k_2,\cdots\}$ are called to be *integral constants* of the counting process. Now we present some definition and theorems which will be required in the next section.

Let $\int_* f(t)dt=\int f(t)dt-C$, where C is an integral constant of $f(t)$. The function $f(t)$ is said to be a *t-zero function* if $[\int_* f(t)dt]_{t=0}=0$.

Definition. The counting process $\{N(t)|t\geq 0\}$ is said to be a *polynomial process (P-process) with intensity function* $g_n(t)$ if

(i) $N(0) = 0$,

(ii) $P\{N(t+h) - N(t) = 1 | N(t) = n\} = g_n(t)h + o(h)$
where $\infty < [\int_* g_n(t)dt]_t = 0 < \infty$,

(iii) $P\{N(t+h) - N(t) \geq 2 | N(t) = n\} = o(h)$ for each $= 0, 1, 2, \cdots$.

Theorem 1. Let $\{N(t) | t \geq 0\}$ be a P-process with intensity function $g_n(t)$. Then

(1) $g_0(t)$ is a t-zero function if and only if $k_0 = 1$,

(2) $g_{n-1}(t)P_{n-1}(t)\exp\Big(\int_* g_n(t)dt\Big)(n \geq 1)$ is a t-zero function if and only if $k_n = 0$.

Definition. The P-process $\{N(t) | t \geq 0\}$ is called to be a strongly P-process if

$$k_0 = 1 \text{ and } k_n = 0 (n \geq 1).$$

Let X be a geometric random variable. Then random variable $Y = X - 1$ is called to be *transformed geometric*.

Definition. The P-process $\{N(t) | t \geq 0\}$ is said to be a *transformed geometric Poisson process* with intensity function $f(t)$ if

(i) $f(0) = 0$,

(ii) $0 \leq f(t) < 1$ for each $t \geq 0$,

(iii) $g_n(t) = (n+1)\frac{df(t)/dt}{1-f(t)}$.

We know that the transformed geometric Poisson process has the intensity function $g_n(t)$ such that $g_i(t) \neq g_j(t)$ for $i \neq j$.

2. Main Result

Let X be a continuous random variable on $[0, \infty)$ and F be a distribution of X such that $F(t) < 1$ for each $t \in R^+$. And let $f(t)$ is a probability density function of X.

Lemma 1. The failure rate function $\lambda(t)$ of X is a t-zero function.

Proof. Since $\lambda(t) = \frac{f(t)}{1-F(t)}$,

$$\begin{aligned} \left[\int_* \lambda(t)dt\right]_{t=0} &= \left[\int_* \frac{f(t)}{1-F(t)}dt\right]_{t=0} \\ &= [-\ln(1-F(t)]_{t=0} \\ &= 0. \end{aligned}$$

Therefore the failure rate function is t-zero.

Theorem 2. Let $\lambda(t)$ be a failure rate function of X. If the counting process $\{N(t)|t \geq 0\}$ satisfies :

(1) $N(0) = 0$,

(2) $P\{N(t+h) - N(t) = 1|N(t) = n\} = (n+1)\lambda(t) + o(h)$,

(3) $P\{N(t+h) - N(t) \geq 2|N(t) = n\} = o(h)$ for each $n = 0, 1, 2, \cdots$.

Then $\{N(t)|t \geq 0\}$ is a transformed geometric Poisson process with intensity function $F(t)$.

Proof. Let $f(t)$ be a probability density function of X and $\lambda(t)$ be a failure rate function of X. Then

$$\lambda(t) = \frac{f(t)}{(1-F(t))}$$

and the intensity function of the process is

$$g_n(t) = (n+1)\frac{f(t)}{1-F(t)}.$$

We obtain

$$\left[\int_* (n+1)\frac{f(t)}{1-F(t)}dt\right]_{t=0} = -(n+1)[\ln(1-F(t))]_{t=0} = 0.$$

Thus $\{N(t)|t \geq 0\}$ is a P-process.

Since $F(t)$ is distribution of X, $F(0) = 0$ and $0 \leq F(t) < \infty$ by assumption. Therefore, $\{N(t)|t \geq 0\}$ is a transformed geometric Poisson process with intensity function $F(t)$.

Definition 1. Let $\{N(t)|t \geq 0\}$ be a transformed geometric Poisson process with intensity function $F(t)$. The P-process $\{N(t)|t \geq 0\}$ is said to be a *transformed* *geometric Poisson process with respect to random variable* X if $F(t)$ is a distribution of X such that $F(0) = 0$ and $F(t) < 1$ for each $t \in R^+$.

Let X_1 denote the time of the first event. Further, for $n \geq 1$, let X_n denote the time btween the $(n-1)$st and n th events.

Theorem 3. Let $\{N(t)|t \geq 0\}$ be a transformed geometric Poisson process with respect to random variable X. Then $F(t)$ is a distribution of X_1 (ie.$X =^d X_1$).

Proof. Let X_1 denote the time of the first event. Since

$$\begin{aligned} P\{N(t) = n\} &= (1 - F(t))^n F(t), \\ P\{X_1 > t\} &= P\{N(t) = 0\} \\ &= 1 - F(t) \\ &= P\{X > t\}. \end{aligned}$$

Therefore, $F(t)$ is a distribution of X_1.

Proposition 4. Let $\{N(t)|t \geq 0\}$ be a transformed geometric Poisson process with intensity function $F(t)$ with respect to random variable X and let $P_n(t) = P\{N(t) = n\}$. Then

(1) $P_0(t)$ is decreasing,

(2) $P_n(t)$ is decreasing on $\left[0, F^{-1}(\frac{1}{n+1})\right)$ and increasing on $\left(F^{-1}(\frac{1}{n+1}), \infty\right)$.

Proof. (1) Since $\{N(t)|t \geq 0\}$ is a transformed geometric Poisson process with intensity function $F(t)$ with respect to random variable X, $P_0(t) = 1 - F(t)$. $P_0(t) = 1 - F(t)$ is decreasing.

(2) Since $P_n(t) = (1 - F(t))^n F(t)$

$$\frac{d}{dt} P_n(t) = n(1 - F(t))^n(-f(t))F(t) + (1 - F(t))^n f(t)$$

Thus $\frac{d}{dt} P_n(t) = 0$ at $F(t) = \frac{1}{n+1}$.

Example. Let X be a Pareto random variable. Then distribution of X is

$$F_X(x) = \begin{cases} 1 - (\frac{k}{x})^a & , x \leq k. \\ 0 & , \text{otherwise}. \end{cases}$$

where $k > 0$ and $a > 0$.
The density function of X is

$$f_X(x) = \frac{ak^a}{x^{a+1}} \qquad (x \geq k > 0).$$

The failure rate function $\lambda(t)$ is

$$\lambda(t) = \frac{f_X(t)}{1 - F_X(t)} = \frac{a}{t}.$$

And we obtain that $P_0(t) = \left(\frac{k}{t}\right)^a$ is decreasing.

$$P_n(t) = \left(\frac{k}{t}\right)^{na} \left\{1 - \left(\frac{k}{t}\right)^a\right\}$$

$$\begin{aligned} P_n'(t) &= \frac{d}{dt}\left[\left(\frac{k}{t}\right)^{na} \left\{1 - \left(\frac{k}{t}\right)^a\right\}\right] \\ &= -nak^{na}t^{-na-1} + (na+a)k^{na+a}t^{-na-a-1}. \end{aligned}$$

Thus

$$P_n'(t) = 0 \text{ at } t = \sqrt[a]{\frac{n+1}{n}}k = F_X{}^{-1}\left(\frac{1}{n+1}\right).$$

Therefore $P_n(t)$ is decreasing on $\left(\sqrt[a]{\frac{n+1}{n}}k, \infty\right)$.

Theorem 5. Let $\{N(t)|t \geq 0\}$ be a transformed geometric Poisson process with respect to Pareto random variable X. Then $\{N(t)|t \geq 0\}$ is a strongly P-process.

Proof. By Lemma 1, $g_0(t) = \lambda(t)$ is t-zero function. Thus $k_0 = 1$.
Since

$$\begin{aligned} \exp\left[\int_* g_n(t)dt\right] &= \exp\left[\int_* \frac{(n+1)a}{t}dt\right] \\ &= \exp[(n+1)a \ \mathrm{ln}t] = t^{na+a}. \end{aligned}$$

Then

$$\begin{aligned} g_{n-1}(t)P_{n-1}(t)\exp\left(\int_* g_n(t)dt\right) &= \left(\frac{na}{t}\right)\left[\left(\frac{k}{t}\right)^{(n-1)a}\left(1 - \left(\frac{k}{t}\right)^a\right)\right]t^{(n+1)a} \\ &= nak^{(n-1)a}t^{2a-1} - nak^n t. \end{aligned}$$

Hence,

$$\Big[\int_* g_{n-1}(t)P_{n-1}(t)\exp\Big(\int_* g_n(t)dt\Big)dt\Big]_{t=0} = \Big[\int_*(nak^{(n-1)a}t^{2a-1} - nak^n t)dt\Big]_{t=0}$$
$$= 0.$$

Thus $g_{n-1}(t)P_{n-1}(t)\exp[\int_* g_n(t)dt]$ is t-zero function. By Theorem 2 in reference 5, $k_n = 0$. Therefore $\{N(t)|t \geq 0\}$ is a strongly P-process.

References

1. Chon, A. C. (1960). Estimating the parameters of modified Poisson distribution, *Journal of the American Statistical Association*, Vol. 55,139-143.

2. Johnson, N. L., S. Kotz and Balakrishnan, N. (1994). *Continuous Univariate dist- ribution-s.* Vol. 1, Second. Edition. John Wiley & Sons, New York.

3. Concul, P. C. (1989). *Generalized Poisson Distributions Properties and Application.* Marcel Dekker, Inc. New York.

4. Concul, P. C. and Shoukri, M. M. (1988). Some chance mechanisms generating the generalized Poisson Probability models, *Amer. J. Math. Management Sci.*, 8(1).

5. Park, J. H. (1997). The counting Process of Which the Intensity Function Depend on States, *The Korean Communications in Statistics*, Vol. 4, No. 1, 281-292.

6. Rao, C. R. and Rubin, H. (1964). On a characterization of the Poisson distribution, *Sankhya, Series A*, Vol. 26, 295-298.

7. Ross, S. D. (1993). *Introduction to probability Models*, Fifth Edition, Academic Press, Inc.