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First-Passage Time Distribution of Discrete Time Stochastic Process with 0-state [1]

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Abstract

We handle the stochastic processes of independent and identically distributed random variables. But random variables are usually dependent among themselves in actual life. So in this paper, we find out a new process not satisfying Markov property. We investigate the probability mass functions and study on the probability of the first-passage time. Also we find out the average frequency of continuous successes in from 0 to n time.

Key Words and Phrases: Markov property, recurrence process, first-passage time distribution.

1. Introduction

A Markov process is a process with the property that, given the value of X_t, the value of $X_s, s > t$, do not depend on the values of $X_u, u < t$; that is, the probability of any particular future behavior of the process, when its present state is known exactly, is not altered by additional knowledge concerning its past behavior. A Markov process having a finite or denumerable state space is called a Markov chain. A Markov process for which all realizations or sample functions $\{X_t, t \in [0, \infty)\}$ are continuous functions is called a diffusion process A discrete time Markov chain $\{X_n\}$ is a Markov stochastic process whose state space is a countable or finite set, and for which $T = (0, 1, 2, \cdots)$. In general, we handle the stochastic processes of independent and identically distributed random variables. But random variables are usually dependent among themselves in actual life. So the objects of this paper are to define a new process not satisfying Markov property and to find the probability mass functions, the first-passage time distributions and the average frequency of continuous successes in from 0 to n time

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2. Formulation of Model and Definition

There are the following meanings in the defined model. Firstly, the concern is put on the frequency of an event happening successively. Secondly, we can see that the model returns to the original state (0 or the previously appointed state) unless the desired success happens before reaching the previously limited successive frequency. In other words, we consider a process comes to stay at 0-state automatically till N-time if continuous successes happen as much as the frequency of M. Here, the probability staying at 0-state from time after arriving at M-state to next N-time is 1. We consider that i-state $(0 \leq i < M)$ is changed to $(i+1)$-state with probability p_i. if succeeded and if not, it is changed to 0-state with probability $1 - p_i$. Hence, the given model can be defined as follows :

Definition 1. A stochastic process $\{X_n, n = 0, 1, 2, \cdots\}$ witha state space $\{0, 1, 2, \cdots M\}$ and a time space $\{0, 1, 2, \cdots\}$ is said to be a recurrence process with 0-state if

(1) $P\{X_0 = 0\} = 1$

(2)
$$\begin{aligned} P\{X_{n+1} = i+1 | X_n = i\} &= P\{X_{n+1} = i+1 | X_n = i, X_q = x(q), 0 \leq q \leq n-1\} \\ &= \begin{cases} p_i, & \text{if } i \neq M \\ 0, & \text{if } i = M \end{cases} \end{aligned}$$

(3) $P\{X_{n+k} = 0 | X_n = M\} = 1$ for $k = 1, 2, \cdots, N$

(4)
$$\begin{aligned} P\{X_{n+1} = 0 | X_n = i\} &= P\{X_{n+1} = 0 | X_n = i, X_q = x(q),\ 0 \leq q \leq n-1\} \\ &= 1 - p_i \text{ for } i \neq M. \end{aligned}$$

That is, M is maximum value of the state space and N represents the time staying at 0-state after arriving at M-state. From the above definition, we can see that the recurrence process with 0-state returns to 0-state with probability $1 - p_i (0 \leq i < M)$ unless the successive events happen as much as the frequency of M . And it returns to 0-state with probability 1 if the successive events happen at M times. From the condition (3), we know that the recurrence process with 0-state does not satisfy Markov property only in the case of $N > 1$. Throughout this chapter, we will consider the case of $p_i = p,\ i = 0, 1, \cdots, M-1$ because of complexity of calculation. We can find the above-defined model in real life. For instance, the following examples belong to this model.

Example 2. If the stock prices of joint-stock corporations listed in stock company have upper price (or lower price) continually for six days,they are going to be suspended in dealings for three days. Unless they have successive upper price, we calculate the frequency of upper price again from the original state. Accordingly we become concerned with the successive frequency in this case.

Example 3. A man who passed the written test in an exam for car license should pass another three kinds of course test. And then he also takes the long-distance test. In these course tests, he has to pass successively without even one fail in it. Unless he passes, he must take a course test again in a few days. So this case belongs to the above model.

3. Probability Mass Functions

In this chapter, we will examine the mathematical results of a recurrence process with 0-state. We investigate each the probability mass functions when separate larger time from smaller time on the standard of time M. After selecting several times, we calculate the probability mass functions.

Theorem 4. Let $\{X_n, n = 0, 1, 2, \cdots\}$ be a recurrence process with 0-state. Then for $0 \leq n \leq M$,

$$P\{X_k = k\} = p^k, \ k = 0, 1, \cdots, M$$
$$P\{X_k = k - j\} = p^{k-j}(1-p), \ k = j, \cdots, M; j = 1, 2, \cdots, M$$
$$P\{X_k = k - j + M\} = 0, k = 0, 1, \cdots, j; j = 0, 1, \cdots, M-1.$$

Proof.
By definition, $P\{X_0 = 0\} = 1$.
In case of $1 \leq n \leq M$,

$$\begin{aligned} P\{X_n = i\} &= \sum_{j=0}^{\infty} P\{X_n = i | X_{n-1} = j\} P\{X_{n-1} = j\} \\ &= \begin{cases} P\{X_n = i | X_{n-1} = i-1\} P\{X_{n-1} = i-1\}, & i \neq 0 \\ \sum_{j=0}^{M} P\{X_n = 0 | X_{n-1} = j\} P\{X_{n-1} = j\}, & i = 0 \end{cases} \\ &= \begin{cases} pP\{X_{n-1} = i-1\}, & i \neq 0 \\ 1-p, & i = 0. \end{cases} \end{aligned}$$

Using the given results, we can calculate $E\{X_n\}$. As it is a successive average frequence of successes from time 0 to n, the result of calculation can be applied as a numerical value forecasting the successful frequency from the present time to the next period.

Theorem 5. If $\{X_n, n = 0, 1, 2, \cdots\}$ is a recurrence process with 0-state, then for $n < M + 1$,

$$E\{X_n\} = \frac{p(1-p^n)}{1-p}.$$

Proof.
In case of $n < M + 1$,

$$\begin{aligned}
E\{X_n\} &= E\{E\{X_n|X_{n-1}\}\} \\
&= \sum_{i=0}^{n} E\{X_n|X_{n-1}=i\}P\{X_{n-1}=i\} \\
&= \sum_{i=0}^{n-2}\sum_{j=0}^{n} jP\{X_n=j|X_{n-1}=i\}P\{X_{n-1}=i\} \\
&\quad + \sum_{j=0}^{n} jP\{X_n=j|X_{n-1}=n-1\}P\{X_{n-1}=n-1\} \\
&\quad + \sum_{j=0}^{n} jP\{X_n=j|X_{n-1}=n\}P\{X_{n-1}=n\} \\
&= \sum_{i=0}^{n-2}\sum_{j=0}^{n} jP\{X_n=j|X_{n-1}=i\}p^i(1-p) \\
&\quad + nP\{X_n=n|X_{n-1}=n-1\}P\{X_{n-1}=n-1\} \\
&= (1-p)\sum_{k=1}^{n-1} kp^k + np^n \\
&= \frac{p(1-p^{n-1})}{1-p} - (n-1)p^n + np^n \\
&= \frac{p(1-p^n)}{1-p}.
\end{aligned}$$

4. First-Passage Time Distributions

In this chapter, we study on the first-passage time distributions at maximum value M in state space. As N is the time staying at 0-state after reaching maximum value M, we consider the first-passage time distributions when $N = 1$. After selecting several times, we calculate the probability mass functions. Define the first-passage time the process hits M by

$$T_M = \min\{n : X_n = M\}.$$

That is,

$$T_M = n \quad \Leftrightarrow \quad X_n = M, \ X_k \neq M \quad \text{for } k = 0, 1, \cdots, n-1.$$

In case of $n \leq 2M$, the first-passage time is n if and only if the state is M at time n . Using p.m.f., we can obtain the following results :

$$\begin{aligned} P\{T_M = n\} &= P\{X_n = M\} \\ &= \begin{cases} 0, & \text{if } n = 0, 1, \cdots, M\text{-}1 \\ p^M, & \text{if } n = M \\ p^M(1-p), & \text{if } n = M\text{+}1, M\text{+}2, \cdots, 2M. \end{cases} \end{aligned}$$

In case of $n \geq 2M + 1$;

$$\begin{aligned} P\{T_M &= 2M+1\} \\ &= P\{X_{2M+1} = M\} - P\{X_M = M|X_0 = 0\}P\{X_{M+1} = 0|X_M = M\} \\ &\qquad \times P\{X_{2M+1} = M|X_{M+1} = 0\} \\ &= p^M(1 - p + p^{M+1}) - p^M p^M \\ &= p^M(1-p)(1-p^M). \end{aligned}$$

From the above contents, we investigate the p.m.f. of the first-passage time at time $2M + 6$. Though the state at time n is M-state, the time n may not be the first-passage time. The process reaches M-state before time n and then can arrive again at M- state on the time n. In other words, there are two instances to be M-state at time $2M+6$. First, it reaches M-state for the first time at time $2M+6$. Second, it reaches again M-state at time $2M + 6$ after once arriving at M-state among the times of $M, M+1, M+2, M+3, M+4$, and $M+5$. So, to get the probability applied to the first instance, we must substract the probability applied to the second from the probability of staying at M- state at time $2M + 6$. Hence, P { reaching M-state for the first time at $2M + 6$ } = P { M-state at time $2M + 6$ } - P { reaching M- state at time $2M + 6$ after once arriving at M-state among the times of $M, M+1, M+2, M+3, M+4,$ and $M+5$}.

$$\begin{aligned} P\{T_M &= 2M+6\} \\ &= P\{X_{2M+6} = M\} - P\{X_M = M|X_0 = 0\}P\{X_{M+1} = 0|X_M = M\} \\ &\qquad \times P\{X_{M+6} = 0|X_{M+1} = 0\}P\{X_{2M+6} = M|X_{M+6} = 0\} \\ &\quad - P\{X_1 = 0|X_0 = 0\}P\{X_{M+1} = M|X_1 = 0\} \\ &\qquad \times P\{X_{M+2} = 0|X_{M+1} = M\}P\{X_{M+6} = 0|X_{M+2} = 0\} \end{aligned}$$

$$
\begin{aligned}
&\quad \times P\{X_{2M+6} = M|X_{M+6} = 0\} \\
&-P\{X_2 = 0|X_0 = 0\}P\{X_{M+2} = M|X_2 = 0\} \\
&\quad \times P\{X_{M+3} = 0|X_{M+2} = M\}P\{X_{M+6} = M|X_{M+3} = 0\} \\
&\quad \times P\{X_{2M+6} = M|X_{M+6} = 0\} \\
&-P\{X_3 = 0|X_0 = 0\}P\{X_{M+3} = M|X_3 = 0\} \\
&\quad \times P\{X_{M+4} = 0|X_{M+3} = M\}P\{X_{M+6} = 0|X_{M+4} = 0\} \\
&\quad \times P\{X_{2M+6} = M|X_{M+6} = 0\} \\
&-P\{X_4 = 0|X_0 = 0\}P\{X_{M+4} = M|X_4 = 0\} \\
&\quad \times P\{X_{M+5} = 0|X_{M+4} = M\}P\{X_{M+6} = 0|X_{M+5} = 0\} \\
&\quad \times P\{X_{2M+6} = M|X_{M+6} = 0\} \\
&-P\{X_5 = 0|X_0 = 0\}P\{X_{M+5} = M|X_5 = 0\} \\
&\quad \times P\{X_{M+6} = 0|X_{M+5} = M\}P\{X_{2M+6} = M|X_{M+6} = 0\} \\
= &\ \ p^M(1-p)(1 - 6p^M + 5p^{M+1} + 7p^{M+2} - 10p^{M+3} + 3p^{M+4}).
\end{aligned}
$$

Simillary, we can obtain the p.m.f. of the first-passage time in process according to the change of the N.

5. Numerical Results

In this chapter, we now consider the probability of hitting the M within time point. And consider the particular case when $M = 3$, $N = 1$ and $p = 0.1(0.1)0.5$. The result is shown Figure 1. Also, if we consider five particular cases when $N = 1, 2, 3$ and $p = 0.1(0.1)0.9$, the results are plotted in Figure 2-6 respectively.

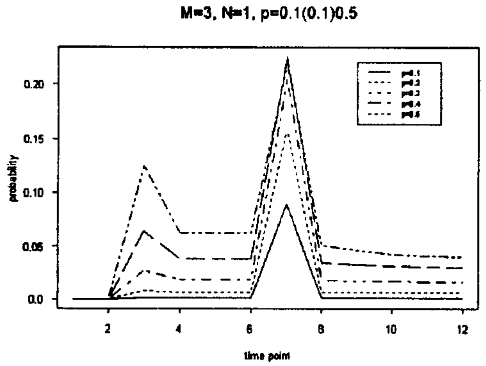

Figure 1

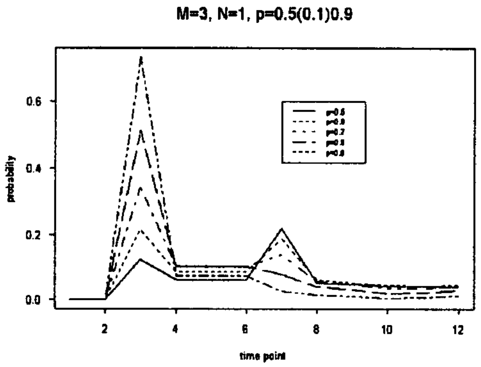

Figure 2

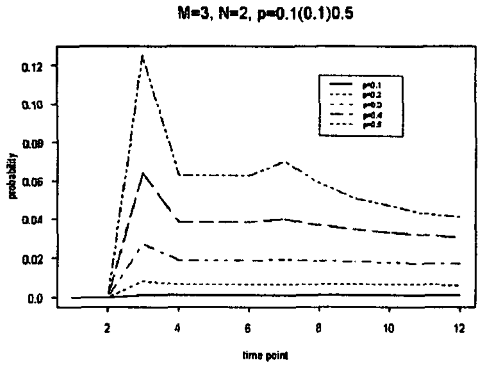

Figure 3

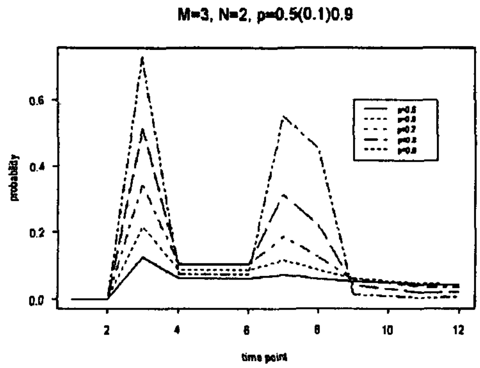

Figure 4

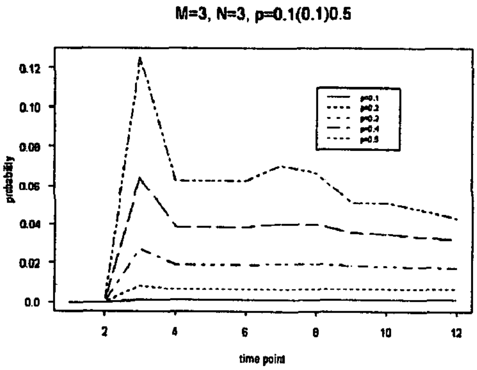

Figure 5

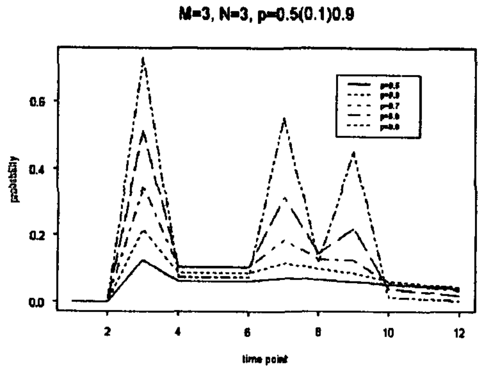

Figure 6

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