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Jackknife Parametric Estimations in a Truncated Arcsine Distribution[1]

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Abstract Maximum likelihood and jackknife estimators of the location and scale parameters and right-tail probability in the truncated arcsine distribution are proposed, and we shall compare the performances of the proposed estimators in terms of bias and mean squared error.

Keywords : Truncated arcsine distribution, Jackknife estimator, MLE.

1. Introduction

A random variable X is said to have a truncated arcsine distribution if its density is of the form

$$f(x\ ;\ \mu,\beta)=\frac{2}{\pi}\frac{1}{\sqrt{\beta^2-(x-\mu)^2}},\quad \mu<x<\mu+\beta, \tag{1.1}$$

where $\mu(\in R)$ and $\beta(>0)$ are reffered as the location and the scale parameters, and denoted it by $X\sim \mathrm{Tarcsin}(\mu,\beta)$. Such random variables arise in the statistical theory of communications(see Lee(1960)) and in the theory of the simple random walk(see Feller(1967)). Here we consider the parametric estimation of the location and scale parameters and right tail probability in a truncated arcsine distribution.

Erdös and Kac(1947), Lee(1960), Middleton(1965), and Feller(1967) studied the property of the arcsine distribution. Norton(1978) and Arnold and Groenveld(1980) derived the characterizations of the arcsine distribution. These characterizations are potentially useful for inference concering whether data arise from a random walk. Such questions are important in consideration of economic dada. Recently, Woo(1996) and Woo, Kim, and Lee(1997) considered properties and the parametric estimation in the arcsine and truncated arcsine distributions.

In this paper, we propose the maximum likelihood(ML) and jackknife estimators for the location and scale parameters and right-tail probability in the truncated

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arcsine distribution and obtain the means and variances for their estimators. Also, we compare the performances of the proposed estimators in terms of bias and mean squared error (MSE) through the Monte Carlo simulation.

2. Parametric Estimations

Let $X_1,\cdots,X_n$ be a simple random sample from Tarcsin(μ,β) and $X_{(1)},\cdots,X_{(n)}$ be the order statistics of the sample.

Note that $X=\mu+\beta\cos Y$ has the truncated arcsine distribution with the location parameter μ and scale parameter β if and only if Y follows an uniform distribution over (0, $\pi/2$) (see Woo(1996)). So let $Y_1,\cdots,Y_n$ be a simple random sample from UNIF$(0,\pi/2)$ and let $Y_{(1)},\cdots,Y_{(n)}$ be the corresponding order statistics. Then $X_{(i)}=\mu+\beta\cos Y_{(n-i+1)}$, $i=1,\cdots,n$.

From Woo et al(1997), means and variances of $X_{(i)}$, $i=1,\cdots,n$, are given as follows ;

$$E\left[X_{(i)}\right]=\mu+\beta(n-i+1)\binom{n}{n-i+1}\sum_{k=0}^{i-1}(-1)^k\binom{i-1}{k}\left(\frac{2}{\pi}\right)^{n-i+1+k}C(n-i+k;1,0)$$
$$\equiv \mu_{i;n},$$

$$Var\left[X_{(i)}\right]=\beta^2\left\{\frac{1}{2}+\frac{n-i+1}{2}\binom{n}{n-i+1}\sum_{k=0}^{i-1}(-1)^k\binom{i-1}{k}\left(\frac{2}{\pi}\right)^{n-i+1+k}C(n+k;2,0)\right. \quad (2.1)$$
$$\left.-\left((n-i+1)\binom{n}{n-i+1}\sum_{k=0}^{i-1}(-1)^k\binom{i-1}{k}\left(\frac{2}{\pi}\right)^{n-i+1+k}C(n+k;1,0)\right)^2\right\}$$
$$\equiv \sigma_{i;n},$$

where

$$C(n;a,b)=\left(\frac{1}{a}\right)^{n+1}\sum_{r=0}^{n}\sum_{s=0}^{r}(-1)^{n-r}s!\binom{n}{r}\binom{r}{s}$$
$$\times\left\{b^{n-r}\left(\frac{a\pi}{2}+b\right)^{r-s}\sin\left(\frac{(a+s)\pi}{2}+b\right)-b^{n-s}\sin\left(\frac{s\pi}{2}+b\right)\right\},$$

and $a\neq 0$, n is non-negative integer.
And covariances for $X_{(i)}$ and $X_{(j)}$, $1\le i<j\le n$, are given by

$$Cov(X_{(i)},X_{(j)}) = \beta^2 \left\{ \frac{n!}{(i-1)!(j-i-1)!(n-j)!} \sum_{k=0}^{j-i-1} \sum_{p=0}^{n-j+k} \sum_{q=0}^{i-1} (-1)^{k+q} p! \right.$$

$$\times \binom{j-i-1}{k} \binom{n-j+k}{p} \binom{i-1}{q} \left(\frac{\pi}{2}\right)^{i-1-q} \left[\frac{1}{2} S(n-i+q-1;2,p\pi/2) \right.$$

$$+ \frac{1}{2(n-i+q)} \left(\frac{\pi}{2}\right)^{n-i+q} \sin\left(\frac{p\pi}{2}\right)$$

$$\left. - (n-j+k)! \sin\left(\frac{(n-j+k)\pi}{2}\right) C(j-i+q-k-1;1,0) \right] \tag{2.2}$$

$$- \left[(n-i+1) \cdot \binom{n}{n-i+1} \sum_{k=0}^{i-1} (-1)^k \binom{i-1}{k} \left(\frac{2}{\pi}\right)^{n-i+1+k} C(n-i+k;1,0) \right]$$

$$\left. \times \left[(n-j+1) \cdot \binom{n}{n-j+1} \sum_{k=0}^{j-1} (-1)^k \binom{j-1}{k} \left(\frac{2}{\pi}\right)^{n-j+1+k} C(n-j+k;1,0) \right] \right\}$$

$$\equiv \sigma_{i,j;n},$$

where

$$S(n;a,b) = \left(\frac{1}{a}\right)^{n+1} \sum_{r=0}^{n} \sum_{s=0}^{r} (-1)^{n-r+1} s! \binom{n}{r} \binom{r}{s}$$

$$\times \left\{ b^{n-r} \left(\frac{a\pi}{2} + b\right)^{r-s} \cos\left(\frac{(a+s)\pi}{2} + b\right) - b^{n-s} \cos\left(\frac{s\pi}{2} + b\right) \right\}$$

and $a \neq 0$, n is non-negative integer.

Here we consider the parametric estimations of the location and scale parameters in the truncated arcsine distribution. In assumed model, the ML estimators for μ and β are given by

$$\hat{\mu} = X_{(1)}, \qquad \hat{\beta} = X_{(n)} - X_{(1)}. \tag{2.3}$$

From (2.1) and (2.2), we can obtain means and variances of $\hat{\mu}$ and $\hat{\beta}$ as follows ;

$$E[\hat{\mu}] = \mu_{1;n}, \qquad Var[\hat{\mu}] = \sigma_{1;n}\,,$$
$$E[\hat{\beta}] = \mu_{n;n} - \mu_{1;n}, \quad Var[\hat{\beta}] = \sigma_{n;n} + \sigma_{1;n} - 2\sigma_{1,n;n}\,. \tag{2.4}$$

From the jackknife technique in Gray et al(1972), the ordinary jackknife estimators of the location and scale parameters are given by

$$J(\hat{\mu}) = \frac{2n-1}{n}X_{(1)} - \frac{n-1}{n}X_{(2)}\,,$$
$$J(\hat{\beta}) = \frac{2n-1}{n}(X_{(n)} - X_{(1)}) - \frac{n-1}{n}(X_{(n-1)} - X_{(2)})\,. \tag{2.5}$$

From (2.1) and (2.2), means and variances of the $J(\hat{\mu})$ and $J(\hat{\beta})$ are given by

$$E[J(\hat{\mu})] = \frac{2n-1}{n}\mu_{1;n} - \frac{n-1}{n}\mu_{2;n}\,,$$
$$E[J(\hat{\beta})] = \frac{2n-1}{n}(\mu_{n;n} - \mu_{1;n}) - \frac{n-1}{n}(\mu_{n-1;n} - \mu_{2;n})\,,$$
$$Var[J(\hat{\mu})] = \left(\frac{2n-1}{n}\right)^2 \sigma_{1;n} + \left(\frac{n-1}{n}\right)^2 \sigma_{2;n} - \frac{2(2n-1)(n-1)}{n^2}\sigma_{1,2;n}\,, \tag{2.6}$$

$$Var[J(\hat{\beta})] = \left(\frac{2n-1}{n}\right)^2 (\sigma_{n;n} + \sigma_{1;n} - 2\sigma_{1,n;n}) + \left(\frac{n-1}{n}\right)^2 (\sigma_{n-1;n} + \sigma_{2;n} - 2\sigma_{2,n-1;n})$$
$$- \frac{2(2n-1)(n-1)}{n^2}(\sigma_{1,2;n} + \sigma_{n-1,n;n} - \sigma_{2,n;n} - \sigma_{1,n-1;n})\,.$$

From the results (2.4) and (2.6), the exact numerical values of biases and MSE's for the ML and jackknife estimators for the location and scale parameters in the truncated arcsine distribution are to have the same trends as those of the simulated results in Table 1 and Table 2 when the sample size equals 10(10)40, $\mu = 0$, and $\beta = 1$. Therefore, the jackknife estimators for the location and scale parameters in the truncated arcsine distribution are more efficient than ML estimators in the sense of bias and MSE.

Table 1. The absolute biases and MSE's for the ML and jackknife estimators of the location parameter μ.

n	$\hat{\mu}$		$J(\hat{\mu})$	
	Bias	MSE	Bias	MSE
10	0.14112	0.03615	0.01723	0.03236
20	0.07302	0.01001	0.00223	0.00995
30	0.04378	0.00480	0.00084	0.00484
40	0.03685	0.00269	0.00032	0.00267

Table 2. The absolute biases and MSE's for the ML and jackknife estimators of the scale parameter β

n	$\hat{\beta}$		$J(\hat{\beta})$	
	Bias	MSE	Bias	MSE
10	0.15935	0.04196	0.00259	0.03489
20	0.07838	0.01096	0.00231	0.01015
30	0.05212	0.00506	0.00145	0.00363
40	0.03834	0.00281	0.00104	0.00269

Where simulations were repeated 5000 times when $\mu = 0$, and $\beta = 1$.

Next , we consider the estimation of the right-tail probability in a truncated arcsine distribution. In a truncated arcsine distribution, the right-tail probability is

$$R(t\,;\mu,\beta) = \frac{2}{\pi}\cos^{-1}\left(\frac{t-\mu}{\beta}\right), \quad \mu < t < \mu+\beta. \tag{2.7}$$

Therefore, the ML estimator for the right-tail probability is given as follows;

$$\hat{R}(t) = \frac{2}{\pi}\cos^{-1}\left(\frac{t-X_{(1)}}{X_{(n)}-X_{(1)}}\right). \tag{2.8}$$

From the jackknife method, the ordinary jackknife estimator for the right-tail probability is defined as

$$J(\hat{R}(t)) = \frac{2(3n-2)}{n\pi}\cos^{-1}\left(\frac{t-X_{(1)}}{X_{(n)}-X_{(1)}}\right) \tag{2.9}$$
$$-\frac{2(n-1)}{n\pi}\left(\cos^{-1}\left(\frac{t-X_{(2)}}{X_{(n)}-X_{(2)}}\right)+\cos^{-1}\left(\frac{t-X_{(1)}}{X_{(n-1)}-X_{(1)}}\right)\right).$$

Finally, using the jackknife estimators for μ and β, we propose the following estimator for the right-tail probability

$$\tilde{R}(t)=\frac{2}{\pi}\cos^{-1}\left(\frac{t-\frac{2n-1}{n}X_{(1)}+\frac{n-1}{n}X_{(2)}}{\frac{2n-1}{n}(X_{(n)}-X_{(1)})-\frac{n-1}{n}(X_{(n-1)}-X_{(2)})}\right). \qquad (2.10)$$

By Monte Carlo simulation, we can obtain the following simulated biases and MSE`s of proposed right-tail probability estimators in the truncated arcsine distribution with the location and scale parameters.

Table 3. The simulated absolute biases and MSE's of proposed estimators for the right-tail probability

	$\hat{R}(t)$		$J(\hat{R}(t))$		$\tilde{R}(t)$	
n	Bias	MSE	Bias	MSE	Bias	MSE
10	0.00126	0.00031	0.00911	0.00099	0.00704	0.00049
20	0.00280	0.00014	0.00868	0.00050	0.00472	0.00024
30	0.00292	0.00010	0.00725	0.00036	0.00409	0.00017
40	0.00221	0.00007	0.00555	0.00025	0.00337	0.00012

Where simulations were repeated 5000 times when $R(t)=0.1$.

From the simulated results, the ML estimator for the right-tail probability in the truncated arcsine distribution with the location and scale parameters. is more efficient than other proposed the right-tail probability estimators in the sense of bias and MSE.

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