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A Random Replacement Model with Minimal Repair[1]

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Abstract In this paper, we consider a random replacement model with minimal repair, which is a generalization of the random replacement model introduced Lee and Lee(1994). It is assumed that a system is minimally repaired when it fails and replaced only when the accumulated operating time of the system exceeds a threshold time by a supervisor who arrives at the system for inspection according to Poisson process. Assigning the corresponding cost to the system, we obtain the expected long-run average cost per unit time and find the optimum values of the threshold time and the supervisor's inspection rate which minimize the average cost.

Keywords : Random Replacement, Minimal Repair, Expected Long-run Average Cost

1. Introduction

A preventive replacement policy may be worthwhile in reducing the cost of operating a stochastically failing system. The most commonly used preventive maintenance policies are the block and age replacement policies. In the block replacement policy, an operating system is preventively replaced by a new one every T units of time and at failure. In the age replacement policy the system is replaced upon failure or at fixed age, whichever comes first. Most of results of the above mentioned policies are well summarized by Barlow and Proschan (1975).

Barlow and Hunter (1960) introduced the block replacement policy with minimal repair at failure. In this policy, a failed system is no longer replaced but instead subject to a minimal repair. A minimal repair puts the system back into its condition just prior to failure. Thus if F is the underlying life distribution of the system, the life of the system being minimally repaired at time t has the survival function given by

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$$\overline{F}(x|t) = \frac{\overline{F}(x+t)}{\overline{F}(t)}, \qquad x > 0$$

Cleroux et al. (1979) proposed the age replacement where failures are corrected by a minimal repair or by replacement. If the cost of repair is less than a fixed constant, then a minimal repair is made, otherwise the system is replaced. These models have been generalized by many authors in many directions. See, for example, Abdel-Hameed(1995), Sheu et al. (1995) and Lee and Lee(1996a,b).

In this paper, we consider the random replacement model introduced by Lee and Lee (1994) and extend this model with minimal repairs. It is assumed that the system is minimally repaired by the repairman who resides in the system when it fails and replaced only by a supervisor who arrives at the system for inspections according to Poisson process with rate λ; the accumulated operating time of the system when he inspects exceeds a threshold time α (>0), he replaces it by a new one, otherwise he does nothing. Both repair and replacement take negligible time.

After assigning costs to each inspection, to each minimal repair, to each replacement and to the system operating in bad state, we obtain the expected long-run average cost $C(\alpha,\lambda)$ for a given threshold time α and an inspection rate λ and discuss the optimal α^* and optimal λ^* which eventually minimize the average cost $C(\alpha,\lambda)$.

2. The expected long-run average cost

Let c_1 denote the cost per inspection of the supervisor, let c_2 denote the expected cost of a minimal repair, let c_3 denote the expected cost of a replacement and let c_4 denote the penalty cost per unit time of the system operating in bad state which means that the accumulated operating time exceeds α. It is assumed that the system has a failure time distribution $F(x)$ with finite mean and has a density $f(x)$. Then the failure rate is $r(x) = f(x)/\overline{F}(x)$ and the cumulative hazard is $R(x) = \int_0^x r(x)dx$, which has a relation $R(x) = -\ln \overline{F}(x)$, where $\overline{F}(x) = 1 - F(x)$. It is further assumed that the failure rate $r(x)$ is differentiable and remains undisturbed by minimal repair.

The points where the actual replacement occurs form an embedded renewal process because the system restarts probabilistically at these points. Let T be the generic random variable denoting the time interval between two successive replacements and $\{N(t), t \geq 0\}$ be the Poisson process of rate λ and $S_1, S_2, \cdots$ the successive arrival times. Then,

$$T \equiv S_{N(\alpha)+1} \tag{1}$$

where ≡ means the equality in distribution. From equation (1) and the memoryless property of the exponential random variable it follows that

$$E[T] = E[S_{N(\alpha)+1}] = \alpha + \frac{1}{\lambda}. \tag{2}$$

Consider the minimal repairs during the cycle T. Let $M(t),\ \ 0 \le t \le T$ denote the number of minimal repairs. Then $\{M(t), 0 \le t \le T\}$ is a nonhomogeneous Poisson process with intensity $r(t)$. (See Block et al. (1985))
It can be easily seen that the expected total cost during a cycle for a given threshold time α and an inspection rate λ is given by

$$C_T(\alpha,\lambda) = E[c_1(N(\alpha)+1)] + c_2 E[M(\alpha)] + c_3 + c_4 E[E_\lambda]$$
$$= c_1(\alpha\lambda + 1) + c_2 R(\alpha) + c_3 + c_4 \frac{1}{\lambda} \tag{3}$$

where E_λ denote the exponential random variable with rate λ.

By applying the Renewal Reward theorem we obtain the expected long-run average cost $C(\alpha,\lambda)$ for given α and λ from equations (2) and (3)

$$C(\alpha,\lambda) = \frac{C_T(\alpha,\lambda)}{E[T]}$$
$$= \frac{c_1(\alpha\lambda+1)\lambda + [c_2 R(\alpha) + c_3]\lambda + c_4}{1+\alpha\lambda} \tag{4}$$

3. Optimal values for α and λ

In this section, we find the conditions under which the average cost $C(\alpha,\lambda)$ has minimum with respect to α and λ, respectively.

Theorem. If the underlying distribution F belongs to the strictly IFR class with $r(0) = 0$, then there exists a unique optimum value α^* which minimizes $C(\alpha,\lambda)$ for a fixed λ.

Proof. Differentiating $C(\alpha,\lambda)$ with respect to α we obtain that

$$\frac{\partial}{\partial\alpha} C(\alpha,\lambda) = \frac{A(\alpha)}{(1+\alpha\lambda)^2},$$

where $A(\alpha) = c_2\lambda^2 \int_0^\alpha x r'(x)dx + c_2\lambda r(\alpha) - (c_3\lambda + c_4)\lambda$.

If the underlying distribution F has a nonconstant increasing failure rate $r(x)$,

$A'(\alpha) = c_2 r'(\alpha)(1+\alpha\lambda)\lambda \geq 0$ so that $A(\alpha)$ is a nondecreasing function of α. Notice that

$$\lim_{\alpha \to 0} A(\alpha) = -(c_3\lambda + c_4)\lambda < 0$$

and

$$\lim_{\alpha \to \infty} A(\alpha) = \infty.$$

It follows that there exists a unique value $\alpha^*(>0)$ which minimizes $C(\alpha,\lambda)$ for a given λ.

Theorem. If $c_1 + c_2 R(\alpha) + c_3 \geq \alpha c_4$, then $C(\alpha,\lambda)$ achieves its minimum value at $\lambda = 0$, otherwise there exists a unique λ^* which minimizes $C(\alpha,\lambda)$ for a given α.

Proof. We differentiate $C(\alpha,\lambda)$ with respect to λ as follows;

$$\frac{\partial}{\partial\lambda} C(\alpha,\lambda) = \frac{B(\lambda)}{(1+\alpha\lambda)^2},$$

where $B(\lambda) = c_1\alpha^2\lambda^2 + 2c_1\alpha\lambda + c_1 + c_2R(\alpha) + c_3 - \alpha c_4$.
Suppose that $c_1 + c_2R(\alpha) + c_3 \geq \alpha c_4$. Since $\lim_{\lambda\to 0} B(\lambda) \geq 0$ and $B'(\lambda) = 2c_1\alpha^2\lambda + 2c_1\alpha \geq 0$ for all $\lambda \geq 0$ it follows that $B(\lambda) \geq 0$ for all $\lambda \geq 0$ and $C(\alpha,\lambda)$ is minimized at $\lambda = 0$. Suppose, now, that $c_1 + c_2R(\alpha) + c_3 < \alpha c_4$, then $\lim_{\lambda\to 0} B(\lambda) < 0$. Since $B(\lambda)$ is a nondecreasing function with $\lim_{\lambda\to\infty} B(\lambda) = \infty$, there exists a unique value λ^* such that $B(\lambda^*) = 0$, which minimizes $C(\alpha,\lambda)$. This optimum value λ^* is given by

$$\lambda^* = -\frac{1}{\alpha} + \sqrt{\frac{c_4\alpha - c_2R(\alpha) - c_3}{c_1\alpha^2}}.$$

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