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Bayesian Approach to the Prediction in the Censored Sample from Rayleigh Population

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Abstract S independent sample $0, 1, 2, \cdots, s-1$ (or stages $0, 1, 2, \cdots, s-1$) are available from the Raleigh population. Procedure for predicting any order statistic in the $(s+1)^{th}$ sample is developed by obtaining the predictive distribution at stage s. Bounds for the sample size at stage s, in order to have the variance at stage s less than that at stage $(s-1)$, are obtained.

Keywords : Bayesian approach, Predictive density function, sampling stage, censored samples, variance

1. Introduction

Prediction problem, which is receiving much attention recently, has been viewed mostly in two directions. One is the classical approach and the other is Bayesian approach. Based on Bayesian approach, Dunsmore (1974) deals with the two sample problem while predicting the observations in the second sample based on the first sample. Lingappaiah(1979,1984) generalizes this to s samples where an observation on the sample is predicted in terms of previous $(s-1)$ samples. Chhikara and Guttman (1982), Sinha(1989), Upadhyay and Pandey(1989) and Nigm and AL-Wahab(1996) suggested the Bayesian approach to the prediction for Gaussian, Lognormal, Exponential and Burr distribution, respectively.

The motivation is to predict when more than two samples are available. Earlier information at any sampling stage is used by considering the posterior distribution at any stage as the prior for the next stage. After obtaining the predictive distribution at a stage, the effect of the sample size and the variance at any stage are discussed.

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2. Prediction density with order statistics

Suppose $(s+1)$ samples are available from the Rayleigh population represented by

$$f(x:\sigma)=\frac{x}{\sigma^2}\exp\left(-\frac{x^2}{2\sigma^2}\right),\quad \sigma>0,\quad x>0. \tag{1}$$

Then the likelihood function under the type II censoring at the r-th failure is

$$L(\sigma|\underline{x})\propto\frac{1}{\sigma^{2r}}\exp\left(-\frac{\sum_{j=1}^{r}x_j^2+(n-r)x_{(r)}^2}{2\sigma^2}\right). \tag{2}$$

Now, an inverse gamma prior distribution for σ is assumed which is given by

$$\pi(\sigma)=\frac{\exp(-\frac{1}{2\beta\sigma^2})}{\Gamma(\alpha)\beta^{\alpha}\sigma^{2(\alpha+1)}}\quad,\quad \alpha,\beta>0,\quad \sigma>0. \tag{3}$$

Then the posterior density of σ given $\underline{X}=\underline{x}$ is

$$\pi(\sigma|\underline{x})=\frac{\left(\beta T^2+2/\beta\right)^{r+\alpha+\frac{1}{2}}}{\Gamma\left(r+\alpha+\frac{1}{2}\right)2^{r+\alpha-\frac{1}{2}}}\exp\left(-\frac{\beta T^2+2}{2\beta\sigma^2}\right), \tag{4}$$

where $T^2=\sum_{1}^{r}x_i^2+(n-r)x_{(r)}^2$.

Now consider the second sample (1^{st} stage) of size n_1. Then the density of $y_1=x_{1(k_1)}$, that is, k_1^{th} order statistic in the sample of size n_1 is given by

$$\begin{aligned} f(y_1|\sigma)=&\frac{1}{B(k_1,n_1-k_1+1)}\frac{y_1}{\sigma^2}\exp\left(-\frac{(n_1-k_1+1)y_1}{2\sigma^2}\right)\\ &\times\left[1-\exp(-\frac{y_1^2}{2\sigma^2})\right]^{k_1-1},\ y_1>0, 1\le k_1<n_1 \end{aligned} \tag{5}$$

From (4) and (5), we have the predictive density function at the first stage as

$$\pi\left(y_1|\underline{x}\right)=\frac{2\beta\left(r+\alpha+\frac{1}{2}\right)y_1\left(\beta T^2+2\right)^{r+\alpha+\frac{1}{2}}}{B\left(k_1,n_1-k_1+1\right)}$$

$$\times\sum_{i=0}^{k_1-1}\binom{k_1-1}{i}(-1)^i\left\{BT^2+2+\left(n_1-k_1+1+i\right)\beta y_1{}^2\right\}^{-\left(r+\alpha+\frac{3}{2}\right)}. \quad (6)$$

Now suppose we take the posterior for the first stage, that is, $f\left(\sigma|\underline{x},y_1\right)$ as the prior for the second stage and considering a similar expression for $y_2=x_{2(k_2)}$ and (5), that is the probability density function of the k_2^{th} order statistic in the sample of size n_2 at the second stage(3^{rd} sample), we have the predictive density at stage 2 as

$$\pi(y_2|\underline{x},y_1)=\frac{2\beta(r+\alpha+\frac{3}{2})y_2\prod_{j=1}^{2}\sum_{i=0}^{k_j-1}\binom{k_j-1}{i}}{\beta(k_2,n_2-k_2+1)\sum_{i=0}^{k_1-1}\binom{k_1-1}{i}}$$

$$\times(-1)^j\frac{\left\{\beta T^2+2+\sum_{j=1}^{2}(n_j-k_j+1+i)\beta y_j^2\right\}^{-(r+\alpha+\frac{5}{2})}}{\left\{\beta T^2+2+(n_1-k_j+1+i)\beta y_1^2\right\}^{-(r+\alpha\frac{3}{2})}} \quad (7)$$

Continuing on this line and considering the posterior at the stage $(s-1)$ as the prior for stage s, we have the predictive density for the s^{th} stage as

$$\pi(y_s|\underline{x},y_1,\cdots,y_{s-1})=\frac{2\beta(r+\alpha+s+\frac{1}{2})y_s\prod_{j=1}^{s}\sum_{i=0}^{k_j-1}\binom{k_j-1}{i}}{B(k_s,n_s-k_s+1)\prod_{j=1}^{s-1}\sum_{i=0}^{k_j-1}\binom{k_j-1}{i}}$$

$$\times(-1)^j\frac{\left\{\beta T^2+2+\sum_{j=1}^{s}(n_j-k_j+1+i)\beta y_j^2\right\}^{-(r+\alpha+s+\frac{3}{2})}}{\left\{\beta T^2+2+\sum_{j=1}^{s-1}(n_j-k_j+1+i)\beta y_j^2\right\}^{-(r+\alpha+s+\frac{1}{2})}} \quad (8)$$

3. Special case

Suppose we set $k_1 = k_2 = \cdots = k_s = 1$, that is, we predicting the first order statistic at the stage in terms of first order statistics at earlier stages, now (8) reduce to

$$\pi(y_s|y_1,\cdots,y_{s-1},\underline{x})$$

$$= \frac{2n_s(r+\alpha+s+\frac{1}{2})\beta y_s(\beta T^2+2+\sum_{j=1}^{s} n_j\beta y_j^2)^{-(r+\alpha+s+\frac{3}{2})}}{(\beta T^2+2+\sum_{j=1}^{s-1} n_j\beta_j^2)^{-(r+\alpha+s+\frac{1}{2})}}$$

$$= \frac{2n_s(r+\alpha+\frac{1}{2})\beta y_s(B_{S-1}^0)^{\nu}}{(B_s^0)^{\nu+1}} \tag{9}$$

where

$$\nu = r+\alpha+s+\tfrac{1}{2}$$

$$B_s^0 = \beta T^2 + 2 + \sum_{j=1}^{s-1} n_j \beta y_j^2$$

From (9), we have the distribution function of y_s as

$$F(y_s|x,y,\cdots,y_{s-1}) = 1 - \left(\frac{B_{S-1}^0}{B_s^0}\right)^{\nu}. \tag{10}$$

A desirable relationship between n_s and n_{s-1} is expressed in terms of the variances $\sigma_s^2, \sigma_{s-1}^2$ and also in terms of a_s and a_{s-1} where $p\{y_s < a_s\} = p\{y_{s-1} < a_{s-1}\} = \delta$. From (9) it follows

$$\sigma_s^2 = \frac{B_{s-1}^0}{n_s} E_s \tag{11}$$

where $E_s = \left[\frac{1}{\nu-1} - \left(\frac{\Gamma(\frac{3}{2})\Gamma(\nu-\frac{1}{2})}{\Gamma(\nu)}\right)^2\right].$

Now form (11), $\sigma_s < \sigma_{s-1}$ implies

$$y_{s-1} < \sqrt{\frac{B^0_{s-2}}{\beta n_{s-1}}\left\{\left(\sqrt{\frac{n_s}{n_{s-1}}C}\right)^2 - 1\right\}} \tag{12}$$

where $C = \dfrac{\dfrac{1}{\nu-2} - \left(\dfrac{\Gamma(\frac{1}{2})\Gamma(\nu-\frac{3}{2})}{\Gamma(\nu-1)}\right)^2}{\dfrac{1}{\nu-1} - \left(\dfrac{\Gamma(\frac{3}{2})\Gamma(\nu-\frac{3}{2})}{\Gamma(\nu)}\right)^2}$.

and from the predictive density of y_{s-1}, (12) gives

$$p\{\sigma_s < \sigma_{s-1}\} = 1 - \left(\sqrt{\frac{n_s}{n_{s-1}}} \cdot C\right)^{-2(\nu-1)} \tag{13}$$

Either from (12) or (13), it follows that

$$n_s > \frac{n_{s-1}}{C}\text{ , whenever } \sigma_s < \sigma_{s-1}.$$

Again from (10), and $p\{y_s < a_s\} = \delta$, it follows

$$a_s = \sqrt{\left\{(1-\delta)^{-1/\nu} - 1\right\}\frac{B^0_{s-1}}{\beta n_s}}. \tag{14}$$

Now using (14), $a_s < a_{s-1}$ implies

$$y_s < \sqrt{\frac{B^0_{s-2}}{n_{s-1}}\left\{(\frac{n_s}{n_{s-1}} \cdot D) - 1\right\}} \tag{15}$$

where $D = \dfrac{(1-\delta)^{-1/\nu} - 1}{(1-\delta)^{-1/\nu} - 1}$.

and again from the predictive density of y_{s-1}, one gets

$$p\{a_s < a_{s-1}\} = 1 - \left(\sqrt{\frac{n_s}{n_{s-1}}} \cdot D\right)^{-2(V-1)} \tag{16}$$

From (15) or (16), it follows that

$$n_s > \frac{n_s}{D}, \quad \text{whenever} \ a_s < a_{s-1}$$

4. Conclusion

The variance can be reduced by a proper choice n_s and higher v. a_s can be treated similarly since as v increase $(1-\delta)^{-1/v}$ decreases. Even through the variance decreases as the number of stages increases, $\sigma_s < \sigma_{s-1}$ depend very much on y_{s-1} and Q_{s-2}^0. As such, the result developed here is mostly a technique of using all the information available, rather than concluding that the prediction becomes sharper as the number of stage increase. In a sense, it is true as can be seem in the relationship between and σ_s^2 and s.

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