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Bootstrap and Delete-d Jackknife Confidence Intervals for Parameters of an Exponential Distribution

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Abstract We introduce several estimators of the location and the scale parameters of the two-parameter exponential distribution, and then compare these estimators by the mean square error (MSE). Using the parametric bootstrap estimators and the delete-d jackknife, we obtain the bootstrap and the delete-d jackknife confidence intervals for the location and the scale parameters and compare the bootstrap confidence intervals with the delete-d jackknife confidence intervals by length and coverage probability through Monte Carlo method.

Key words: Bootstrap, Confidence interval, Consistency, Delete-d jackknife,

1. Introduction

In life testing research, the simplest and the most widely exploited model is the two-parameter exponential distribution with p.d.f.

$$f(x;\theta,\sigma) = \frac{1}{\sigma}\exp\left(-\frac{x-\theta}{\sigma}\right),\ 0<\theta<x, 0<\sigma, \tag{1.1}$$

where θ and σ are the location and the scale parameters, respectively.

Many authors have utilized an exponential distribution because of its wide applicability in statistical inferences and reliability engineering. Epstein and Sobel (1954) published a paper that presented the maximum likelihood estimators (MLE) of the scale and the location parameter in the two-parameter exponential distribution. Singh et al. (1993) studied the estimation of the parameters of an exponential distribution when the parameter space is restricted.

The bootstrap method, introduced by Efron (1979), is resampling technique and a very general method to create measures of uncertainty and bias, in particular at parameter estimation from independent identically distributed variables.

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Efron (1981, 1987) has introduced and refined the percentile method of using bootstrap calculations to set approximate confidence limits for parameters. These refinements of the percentile method are the bias corrected (BC) percentile method and the accelerated bias-corrected (BC_a) percentile method. The bootstrap method and other methods for assessing statistical accuracy are summarized by Efron and Tishirani (1993). Recently, the theoretical properties of the jackknife and bootstrap methods are summarized by Shao and Tu (1995). Kang and Cho (1997) proposed the nonparametric bootstrap confidence intervals after bias correcting. Kang and Cho (1997) also obtained the bootstrap confidence intervals for the reliability function of an exponential distribution.

The jackknife is a well-known method for bias reduction and robust interval estimation. The jackknife technique is based upon a suggestion of Quenouille (1956). The delete-d jackknife was proposed and studied by Wu (1986, 1990), and Shao and Wu (1989).

In this paper, we will compare the estimators of parameters by the MSE and obtain the parametric bootstrap estimator of parameters. Using the proposed bootstrap estimators and the delete-d jackknife, we obtain the bootstrap and the delete-d jackknife confidence intervals for parameters and compare the bootstrap with the delete-d jackknife confidence intervals through Monte Carlo method, and summarize the numerical results.

2. Bootstrap and delete-d jackknife confidence intervals

Let $X_1, X_2, \ldots, X_n$ be a random sample from the two-parameter exponential distribution (1.1) and let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ be the corresponding order statistics. In the two-parameter exponential distribution, it is well known that the MLEs of σ and θ are $\hat{\sigma}_{MLE} = \Sigma_{i=1}^{n}\left(X_i - X_{(1)}\right)/n$ and $\hat{\theta}_{MLE} = X_{(1)}$. The MSEs of $\hat{\sigma}_{MLE}$ and $\hat{\theta}_{MLE}$ are given by

$$MSE\left(\hat{\sigma}_{MLE}\right) = \sigma^2/n,$$

$$MSE\left(\hat{\theta}_{MLE}\right) = 2\sigma^2/n^2 .$$

The uniformly minimum variance unbiased estimators (UMVUEs) of σ and θ are given by

$$\hat{\sigma}_U = \sum_{i=1}^{n}\left(X_i - X_{(1)}\right)/(n-1),$$

$$\hat{\theta}_U = nX_{(1)}/(n-1) - \sum_{i=1}^{n} X_i/n(n-1).$$

The MSEs of UMVUEs $\hat{\sigma}_U$ and $\hat{\theta}_U$ are

$$MSE(\hat{\sigma}_U) = \sigma^2/(n-1),$$

$$MSE(\hat{\theta}_U) = \sigma^2/n(n-1).$$

Singh et al. (1993) proposed the minimum risk estimator (MRE) of the location parameter which has minimum risk among the class of the estimators of the form $c_1 X_{(1)} + c_2 \overline{X}$ (c_1 and c_2 are constants) as follows;

$$\hat{\theta}_M = (n+1)X_{(1)}/n - \sum_{i=1}^{n} X_i/n^2.$$

The MSE of $\hat{\theta}_M$ is $MSE(\hat{\theta}_M) = (n+1)\sigma^2/n^3$.

The parametric bootstrap method algorithm is the following. From the MSEs of the parametric estimators, we know that the estimators $\hat{\theta}_M$ and $\hat{\sigma}_{MLE}$ are more efficient than the other estimators of the location and the scale parameters in terms of the MSE. So we select B independent bootstrap data set $\boldsymbol{X}^{*1}, \boldsymbol{X}^{*2}, \ldots, \boldsymbol{X}^{*B}$ which are new generated from $F(\hat{\theta}_M, \hat{\sigma}_{MLE})$ and evaluate the bootstrap replications corresponding to each bootstrap sample $(b = 1, 2, \ldots, B)$, $\hat{\theta}^*_M(b) = \hat{\theta}_M(\mathbf{X}^{*b})$ and $\hat{\sigma}^*_{MLE}(b) = \hat{\sigma}_{MLE}(\boldsymbol{X}^{*b})$.

Theorem. The bootstrap estimators $\hat{\sigma}^*_{MLE}$ and $\hat{\theta}^*_M$ are consistent estimators of the scale and the location parameters, respectively.

Proof. For arbitrary positive ε,

$$P(|\hat{\sigma}^*_{MLE} - \hat{\sigma}_{MLE}| \geq \varepsilon) \leq \frac{E[\hat{\sigma}^*_{MLE} - \hat{\sigma}_{MLE}]^2}{\varepsilon^2}$$

$$= \frac{E\left[E\left[(\hat{\sigma}^*_{MLE} - \hat{\sigma}_{MLE})^2 | X_1, X_2, \cdots, X_n\right]\right]}{\varepsilon^2}$$

$$= \frac{E\left[\hat{\sigma}^2_{MLE}\right]}{n\varepsilon^2}$$

$$= \frac{(n-1)\sigma^2}{n^2\varepsilon^2} \rightarrow 0, \text{ as } n \rightarrow \infty$$

and

$$P\left(\left|\hat{\theta}_M^* - \hat{\theta}_M\right| \geq \varepsilon\right) \leq \frac{E\left[\hat{\theta}_M^* - \hat{\theta}_M\right]^2}{\varepsilon^2}$$

$$= \frac{E\left[E\left[\left(\hat{\theta}_M^* - \hat{\theta}_M\right)^2 \mid X_1, X_2, \cdots, X_n\right]\right]}{\varepsilon^2}$$

$$= \frac{(n+1)E\left[\hat{\sigma}_{MLE}^2\right]}{n^3\varepsilon^2}$$

$$= \frac{\left(n^2-1\right)\sigma^2}{n^4\varepsilon^2} \to 0, \text{ as } n \to \infty.$$

Therefore, $\hat{\sigma}_{MLE}^*$ and $\hat{\theta}_M^*$ converge in probability to σ and θ, respectively. This completes the proof.

Using the parametric bootstrap estimators of parameters, we obtain the parametric bootstrap confidence intervals for parameters. Let $\hat{G}$ be the cumulative distribution function of $\hat{\theta}_M^*$. The bootstrap percentile method (Efron, (1981)) gives the following $100(1-2\alpha)\%$ percentile interval for θ;

$$\left[\hat{\theta}_{\%,lo}, \hat{\theta}_{\%,up}\right] = \left[\hat{G}^{-1}(\alpha), \hat{G}^{-1}(1-\alpha)\right].$$

Since by definition $\hat{G}^{-1}(\alpha) = \hat{\theta}_M^{*(\alpha)}$, the $100 \cdot \alpha$ th percentile of the bootstrap distribution, the percentile interval (PI) is given by

$$\left[\hat{\theta}_{\%,lo}, \hat{\theta}_{\%,up}\right] = \left[\hat{\theta}_M^{*(\alpha)}, \hat{\theta}_M^{*(1-\alpha)}\right]. \tag{2.1}$$

From the assumption that

$$T = \frac{\hat{\theta}_M - \theta}{s\hat{e}(\hat{\theta}_M)} \approx t_{n-1},$$

where t_{n-1} represents the Student's t distribution with $n-1$ degrees of freedom and $s\hat{e}(\hat{\theta}_M)$ is the bootstrap estimator of standard error of $\hat{\theta}_M$, the Student's t interval(STI) is given by

$$\left[\hat{\theta}_{\%,lo}, \hat{\theta}_{\%,up}\right] = \left[\hat{\theta}_M - t_{n-1}^{(1-\alpha)} \cdot s\hat{e},\ \hat{\theta}_M - t_{n-1}^{(\alpha)} \cdot s\hat{e}\right], \tag{2.2}$$

where $t_{n-1}^{(\alpha)}$ is the $100 \cdot \alpha$ th percentile of the Student's t distribution with $n-1$

The bias-corrected (BC) and accelerated bias-corrected (BC_a) intervals are a substantial improvement over the percentile interval in both theory and practice. The $100(1-2\alpha)\%$ BC (Efron, (1981)) and BC_a (Efron, (1987)) intervals are given by

$$\left[\hat{\theta}_{\%,lo}, \hat{\theta}_{\%,up}\right] = \left[\hat{\theta}_M^{*(\Phi(2\hat{z}_0+z^{(\alpha)}))}, \hat{\theta}_M^{*(\Phi(2\hat{z}_0+z^{(1-\alpha)}))}\right] \tag{2.3}$$

and

$$\left[\hat{\theta}_{\%,lo}, \hat{\theta}_{\%,up}\right] = \left[\hat{\theta}_M^{*(\alpha_1)}, \hat{\theta}_M^{*(\alpha_2)}\right], \tag{2.4}$$

where

$$\alpha_1 = \Phi\left(\hat{z}_0 + \frac{\hat{z}_0 + z^{(\alpha)}}{1-\hat{a}(\hat{z}_0 + z^{(\alpha)})}\right)$$

$$\alpha_2 = \Phi\left(\hat{z}_0 + \frac{\hat{z}_0 + z^{(1-\alpha)}}{1-\hat{a}(\hat{z}_0 + z^{(1-\alpha)})}\right)$$

and $\Phi(\cdot)$ is the standard normal cumulative distribution function, and $z^{(\alpha)}$ is the $100 \cdot \alpha$ th percentile point of a standard normal distribution. The value of the bias-correction $\hat{z}_0$ is obtained directly from the proportion of the bootstrap replications less than the original estimator $\hat{\theta}_M$,

$$\hat{z}_0 = \Phi^{-1}\left(\frac{\#\left\{\hat{\theta}_M^*(b) < \hat{\theta}_M\right\}}{B}\right),$$

where Φ^{-1} is the inverse function of a standard normal cumulative distribution function. There are various ways to compute the acceleration $\hat{a}$. We calculate $\hat{a}$ by jackknife method. Let $\mathbf{X}^{-i}$ be the original sample with the i th point x_i deleted, let $\hat{\theta}_M^{-i} = \hat{\theta}_M(\mathbf{X}^{-i})$, and define $\hat{\theta}(\cdot) = \sum_{i=1}^n \hat{\theta}_M^{-i}/n$. The acceleration is given by

$$\hat{a} = \frac{\sum_{i=1}^n \left(\hat{\theta}_{(\cdot)} - \hat{\theta}_M^{-i}\right)^3}{6\left[\sum_{i=1}^n \left(\hat{\theta}_{(\cdot)} - \hat{\theta}_M^{-i}\right)^2\right]^{3/2}}.$$

The Student's t interval (STI) and accelerated bias-corrected (BC_a) interval provide accurate confidence sets. However, these methods are not always practical: the STI requires a good variance estimator and the BC_a requires the estimation of the acceleration constant a. Shao and Tu (1995) introduce the hybrid bootstrap (HB) method that is convenient to use. The $100(1-2\alpha)\%$ HB intervals is given by

$$\left[\hat{\theta}_{\%,lo}, \hat{\theta}_{\%,up}\right] = \left[2\hat{\theta}_M - \hat{G}^{-1}(\alpha),\ 2\hat{\theta}_M - \hat{G}^{-1}(1-\alpha)\right]. \tag{2.5}$$

Using the delete-d jackknife, we repeatedly compute and use the statistics of the form $\hat{\theta}_{r,s} = \hat{\theta}_r(X_i, i \in s^c)$, where s is a subset of $\{1, \ldots, n\}$ with size d, s^c is the complement of s, d is an integer depending n, $1 \le d \le n$, and $r = n - d$. Note that for given statistic $\hat{\theta}_M, \hat{\theta}_{r,s}$ is the same statistic but is based on r observations that are obtained by removing $\{X_i, i \in s\}$ from original data set. The cumulative jackknife histogram of $\hat{\theta}_{r,s}$ is given by

$$\hat{H}(x) = \frac{\sum_{s \in S} I\{\hat{\theta}_{r,s} \le x\}}{N},$$

where S is the collection of all the subsets of $\{1, \ldots, n\}$ that have size d and $N = \binom{n}{d}$ is the total number of subsets in S. Using the cumulative jackknife histogram of $\hat{\theta}_{r,s}$, we obtain the $100(1-2\alpha)\%$ delete-d jackknife (D-dJ) intervals as follows;

$$\left[\hat{\theta}_{\%,lo}, \hat{\theta}_{\%,up}\right] = \left[\left(1+\sqrt{\frac{n-d}{d}}\right)\hat{\theta}_M - \sqrt{\frac{n-d}{d}}\hat{H}^{-1}(\alpha),\right. \tag{2.6}$$

$$\left.\left(1+\sqrt{\frac{n-d}{d}}\right)\hat{\theta}_M - \sqrt{\frac{n-d}{d}}\hat{H}^{-1}(1-\alpha)\right].$$

3. Simulated results

Similarly, we can obtain the above intervals for the scale parameter σ. From (2.1) to (2.6), we calculate the numerical values of the bootstrap and the delete-d jackknife confidence intervals for the location parameters for sample size $n = 6(4)10$ (based on 1000 Monte Carlo runs and B=2,000) when the location parameter $\theta = 1$ and the scale parameter $\sigma = 0.5(0.5)1.5$, and then obtain the average length (AL) of approximate confidence intervals as follows;

$$AL = \sum_{i=1}^{1000} \frac{\left(\hat{\theta}_{\%,up}(i) - \hat{\theta}_{\%,lo}(i)\right)}{1000}$$

and the percentage of trials that the indicated interval missed the true value on the and the percentage of trials that the indicated interval missed the true value on the left (*%Miss Left*) or right (*%Miss Right*) side, and coverage probability (CP) of approximate confidence intervals. These values of the scale and the location parameters are given in Table 1 and Table 2, respectively.

From Table1, we obtain the following results;

1). In sample size $n = 6$, the D-3J confidence interval has the most accurate CP but the longest AL, the PI and the HB confidence intervals are better than the other approximate confidence intervals in terms of AL and the HB confidence interval is better than the PI in terms of CP.

2). In sample size $n = 10$, the D-5J confidence interval has the most accurate CP but the longest AL, the D-1J confidence interval has the shortest AL but the lowest CP. The PI and the HB confidence interval are shorter than the other bootstrap confidence intervals in terms of AL, and the HB confidence interval is better than the PI in terms of CP. So, we know that the HB confidence interval for the location parameter provide good interval estimation.

From Table 2, we obtain the following results;

1). In sample size $n = 6$, the BC_a confidence interval has the most accurate CP, but the longest AL, the PC and the HB confidence interval are shorter than the other bootstrap confidence intervals in terms of AL, and the HB confidence interval is better than the PC in terms of CP. The D-1J confidence interval has the shortest AL but small CP.

2). In sample size $n = 10$, the BC_a confidence interval has the most accurate CP, but the longest AL, the PC and the HB confidence intervals are better than the other bootstrap confidence intervals in terms of AL, and the HB confidence interval is better than the PI in terms of CP. The D-3J confidence interval has the shortest AL but small CP.

3). The D-3J confidence interval for the scale parameter is better than the PI in terms of AL and CP.

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Table 1. Comparison of approximate confidence intervals for the location parameter ($\sigma = 0.5$, $\theta = 1.0$)

n	Method	AL	%Miss Left	%Miss Right	CP(90%)
6	PI	.23452	210	0	790
	STI	.30431	112	0	888
	BC	.25524	226	0	774
	BC_a	.23600	210	0	790
	HB	.23452	107	0	893
	D-1J	.28535	146	110	744
	D-2J	.25262	169	0	831
	D-3J	.39324	72	0	928
10	PI	.14340	209	0	791
	STI	.17133	103	0	897
	BC	.17312	233	0	767
	BC_a	.15144	215	0	785
	HB	.14340	90	0	910
	D-1J	.04546	355	253	392
	D-2J	.15075	170	43	787
	D-3J	.23145	62	33	905
	D-5J	.23981	63	0	937

($\sigma = 1.0$, $\theta = 1.0$)

n	Method	AL	%Miss Left	%MissRight	CP(90%)
6	PI	.46131	225	0	775
	STI	.59867	122	0	878
	BC	.50156	244	0	756
	BC_a	.46393	226	0	774
	HB	.46131	118	0	882
	D-1J	.56194	150	104	746
	D-2J	.49647	180	5	815
	D-3J	.77433	78	1	921

10	PI	.28435	204	0	796
	STI	.34028	111	0	889
	BC	.34328	234	0	766
	BC_a	.30080	213	0	787
	HB	.28435	92	0	908
	D-1J	.08970	343	277	380
	D-2J	.30254	171	68	761
	D-3J	.45812	84	49	867
	D-5J	.47605	74	0	926

$(\sigma = 1.5, \theta = 1.0)$

n	Method	AL	%Miss Left	%MissRight	CP(90%)
6	PI	.67403	230	0	770
	STI	.87449	118	0	882
	BC	.73232	256	0	744
	BC_a	.67566	233	0	767
	HB	.67403	116	0	884
	D-1J	.83605	147	93	760
	D-2J	.73225	170	5	825
	D-3J	1.13535	87	1	912
10	PI	.43175	190	0	810
	STI	.51673	111	0	889
	BC	.52163	221	0	779
	BC_a	.45724	201	0	799
	HB	.43175	94	0	906
	D-1J	.13837	345	262	393
	D-2J	.45390	159	49	792
	D-3J	.69529	61	33	906
	D-5J	.71336	64	0	936

Table 2. Comparison of approximate confidence intervals for the scale parameter $(\sigma = 0.5, \theta = 1.0)$

n	Method	AL	%Miss Left	%MissRight	CP(90%)
6	PI	.50952	0	341	659
	STI	.63922	0	245	755
	BC	.68545	39	129	832
	BC_a	.74496	45	106	849
	HB	.50952	10	281	709
	D-1J	.50275	67	245	688
	D-2J	.52624	82	199	719
	D-3J	.50471	123	202	675
10	PI	.43721	2	263	735
	STI	.49381	1	202	797
	BC	.52073	34	95	871
	BC_a	.56694	42	70	888
	HB	.43721	12	209	779
	D-1J	.44346	79	154	767
	D-2J	.46649	38	156	806
	D-3J	.42738	78	154	768
	D-5J	.43431	58	160	782

$(\sigma = 1.0, \theta = 1.0)$

n	Method	AL	%Miss Left	%MissRight	CP(90%)
6	PI	.99895	0	360	640
	STI	1.25348	0	242	758
	BC	1.34420	28	127	845
	BC_a	1.46801	32	114	854
	HB	.99895	6	280	714
	D-1J	.98049	53	240	707
	D-2J	1.02127	77	207	716
	D-3J	.97598	108	200	692

10	PI	.86529	1	280	719
	STI	.97694	0	213	787
	BC	1.03250	32	118	850
	BC_a	1.11969	37	92	871
	HB	.86529	9	223	768
	D-1J	.86376	87	174	739
	D-2J	.91619	39	166	795
	D-3J	.84078	90	169	741
	D-5J	.85394	76	178	746

($\sigma = 1.5$, $\theta = 1.0$)

n	Method	AL	%Miss Left	%MissRight	CP(90%)
6	PI	1.46182	0	355	645
	STI	1.83360	0	268	732
	BC	1.96513	28	139	833
	BC_a	2.13272	31	113	856
	HB	1.46182	3	292	705
	D-1J	1.42765	40	261	699
	D-2J	1.50516	63	233	704
	D-3J	1.43560	114	227	659
10	PI	1.31605	0	257	743
	STI	1.48650	0	197	803
	BC	1.56961	38	104	858
	BC_a	1.70882	46	88	866
	HB	1.31605	8	215	777
	D-1J	1.33889	87	147	766
	D-2J	1.41283	44	148	808
	D-3J	1.30436	94	151	755
	D-5J	1.31761	66	156	778