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Bayesian Reliability Estimation for a Two-unit Hot Standby System

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Abstract we shall propose some Bayes estimators and some generalized maximum likelihood estimators for reliability of a two-unit hot standby system with perfect switch based upon a complete sample of failure times observed from the exponential model and compare the performances of the proposed estimators in terms of mean squared error.

Keywords : Hot standby system, Bayes estimator, Generalized MLE.

1. Introduction

The two-unit standby redundant system configuration is a form of paralleling where only one component is in operation ; if the operating component fails, then another component is brought into operation, and the redundant configuration continues to function. Depending failure characteristic, standby redundancy is classified into three types. Hot standby system, where each component has the same failure rate regardless of whether it is standby or in operation ; Cold standby system, where components do not fail when they are in standby ; Warm standby system, where a standby component can fail but it has a smaller failure rate than the principal component. Here we consider the problem of estimation for a two-unit hot standby system reliability by using the Bayesian approach.

Reliability computations for a two-unit standby redundant systems with constant failure rate are found by Osaki and Nakagawa(1971). Fujii and Sandoh(1984) considered the Bayesian estimation for reliability of a two-unit hot standby redundant system. Kaput and Garg(1990) considered the technique of Markow renewal process to obtain various reliability measures for a two-unit standby system with perfect switch and Shen and Xie(1991) considered the effect of standby redundance at the system and the component level.

In this paper, we shall propose some Bayes estimators and some generalized maximum likelihood estimators(MLE) for reliability of a two-unit hot standby

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system with perfect switch based upon a complete sample of failure times observed from the exponential model and compare the performances of the proposed estimators in terms of mean square error(MSE) through the Monte Carlo simulation.

2. Bayesian Reliability Estimation

We consider an exponential distribution of lifetime governed by the probability density function

$$f(t|\lambda)=\begin{cases}\lambda e^{-\lambda t}, & 0<t<\infty \\ 0, & \text{otherwise.}\end{cases} \quad (2.1)$$

Many authors have utilized the exponential distribution because of its wide applicability in statistical inferences and reliability engineering(Saunders & Mann (1985) and Bain & Engelhart (1987)).

Here we shall consider the Bayesian approach of the estimation for the two-unit hot standby system reliability with perfect switch in an exponential distribution. In the two-unit hot standby system with perfect switch, we shall assume the following :

1. The system consists of two independent and identically distributed units and a switch.
2. One unit serves as a hot standby when the other is in use.
3. The switch is instantaneous when the one in use fails.
4. The times to failure of both units in use and standby are independent and exponentially distributed with the failure rate λ.
5. The unit and the switch are independent.
6. The switch is failure free.

Then the reliability for a two-unit hot standby system with perfect switch at specified mission time t_0 is given by

$$R(t_0)=e^{-\lambda t_0}(1+\lambda t_0), \quad t_0>0. \quad (2.2)$$

Most Bayes estimators of reliability derived so far were based upon the priors on the unknown parameters which are related to the reliability than the reliability itself. Hence we shall consider uniform, gamma and truncated gamma priors on the failure rate λ in the exponential distribution. Also we shall use the squared error loss function to estimate the system reliability.

Let $T_1,\cdots,T_n$ be a simple random sample from an exponential distribution with failure rate λ and $T=\sum_{k=1}^{n}T_k$ be the total test time.

Assume that the failure rate Λ has a uniform prior distribution given by

$$\pi(\lambda;\lambda_1)=\begin{cases}\dfrac{1}{\lambda_1}, & 0<\lambda<\lambda_1\\ 0, & \text{otherwise},\end{cases} \tag{2.3}$$

denoted by $\text{UNIF}(0,\lambda_1)$, where λ_1 is assumed to be known

Then the posterior distribution of the failure rate Λ given the total test time t is given as follows :

$$g(\lambda|t;\lambda_1)=\frac{t^{n+1}}{\Gamma(n+1,\lambda_1 t)}\lambda^n e^{-\lambda t},\quad 0<\lambda<\lambda_1, \tag{2.4}$$

where $\Gamma(a,z)$ represents the standard incomplete gamma function.

Since the Bayes estimator of the system reliability is found by taking the expectation of reliability function (2.2) with respect to the posterior distribution (2.4), under the squared error loss function the Bayes estimator $\hat{R}_U(t_0)$ for the system reliability $R(t_0)$ is given by

$$\hat{R}_U(t_0)=\frac{1}{\Gamma(n+1,\lambda_1 T)}\left(\frac{T}{T+t_0}\right)^{n+1}\{\Gamma(n+1,\lambda_1(T+t_0)) + \frac{t_0}{(T+t_0)}\cdot\Gamma(n+2,\lambda_1(T+t_0))\}. \tag{2.5}$$

Now we consider the generalized MLE which is the largest mode of the posterior density function of Λ given the total test time t. Although such an estimator may not be a Bayes estimator for any standard loss function, it is a reasonable estimator, as it measure the location of the posterior distribution analogous to the posterior mean and median. Therefore, maximizing the posterior distribution of Λ in (2.4) with respect to λ, we can obtain the generalized MLE for the failure rate λ as follows :

$$\tilde{\lambda}_U=\frac{n}{T}. \tag{2.6}$$

Hence, under the uniform prior distribution on Λ, the generalized MLE $\tilde{R}_U(t_0)$ for the system reliability $R(t_0)$ is

$$\tilde{R}_U(t_0)=e^{\tilde{\lambda}_U(t_0)}(1+\tilde{\lambda}_U t_0). \tag{2.7}$$

Suppose that a prior distribution for the failure rate Λ is the gamma distribution with known α and β given by

$$\pi(\lambda:\alpha,\beta)=\frac{1}{\Gamma(\alpha)\beta^{\alpha}}\lambda^{\alpha-1}e^{-\lambda/\beta},\quad 0<\lambda<\infty. \tag{2.8}$$

denoted by GAM(α,β).

The most widely used prior distribution for failure rate L is the gamma distribution. The main reason for this acceptability is the mathematical tractability resulting from the fact that the gamma distribution is the nature conjugate prior distribution. Such authors as Apostolakis and Mosleh (1979) and Grohowski, Hausman, and Lamberson (1976), and others have found the gamma distribution to be sufficiently versatile for practical reliability applications.

According to Bayes theorem, the posterior distribution of Λ given the total test time t is given by

$$g(\lambda|t;\alpha,\beta)=\frac{1}{\Gamma(\alpha+n)(\frac{\beta}{\beta t+1})^{\alpha+n}}\lambda^{\alpha+n-1}e^{-\lambda(\frac{\beta+1}{\beta})},\quad 0<\lambda<\infty, \tag{2.9}$$

which is recognized as a GAM($\alpha+n,\beta/\beta t+1$) distribution.

Under the squared error loss function, the Bayes estimator $\hat{R}_G(t_0)$ for the system reliability $R(t_0)$ is given by

$$\hat{R}_G(t_0)=(\frac{\beta T+1}{\beta T+\beta t_0+1})^{\alpha+n}\{1+\frac{t_0\beta(\alpha+n)}{\beta T+\beta t_0+1}\}. \tag{2.10}$$

Maximizing the posterior distribution of Λ in (2.9) with respect to λ, we can obtain the generalized MLE for λ as follows :

$$\tilde{\lambda}_G=\frac{\beta(\alpha+n-1)}{\beta T+1}. \tag{2.11}$$

Hence, under the gamma prior distribution on Λ, the generalized MLE $\tilde{R}_G(t_0)$ for the system reliability $R(t_0)$ is

$$\tilde{R}_G(t_0)=e^{-\tilde{\lambda}_G t_0}(1+\tilde{\lambda}_G t_0). \tag{2.12}$$

Now, we shall consider the truncated gamma prior distribution for the failure rate Λ with known α, β and λ_1 given by

$$\pi(\lambda;\alpha,\beta,\lambda_1)=\frac{1}{\Gamma(\alpha,\lambda_1/\beta)\beta^{\alpha}}\lambda^{\alpha-1}e^{-\lambda/\beta},\quad 0<\lambda<\lambda_1, \tag{2.13}$$

denoted by TGAM(α,β).

According to Bayes theorem, the posterior distribution of Λ given the total test time t is given by

$$g(\lambda|t;\alpha,\beta,\lambda_1)=\frac{\lambda^{\alpha+n-1}e^{-\lambda(\frac{\beta t+1}{\beta})}}{\Gamma(\alpha+n,\lambda_1(\frac{\beta t+1}{\beta}))(\frac{\beta}{\beta t+1})^{\alpha+n}},\quad 0<\lambda<\lambda_1, \tag{2.14}$$

which is a TGAM($\alpha+n,\beta/\beta t+1$) distribution.

Therefore the Bayes estimator $\hat{R}_{TG}(t_0)$ of the system reliability $R(t_0)$ under the squared error loss function is given by

$$\hat{R}_{TG}(t_0)=\frac{1}{\Gamma(\alpha+n,\lambda_1(T+1/\beta))}(\frac{\beta T+1}{\beta T+\beta t_0+1})^{\alpha+n}\{\Gamma(\alpha+n,\lambda_1(T+t_0+\frac{1}{\beta}))$$
$$+\frac{t_0\beta}{(\beta T+\beta t_0+1)}\Gamma(\alpha+n+1,\lambda_1(T+t_0+1/\beta))\}. \tag{2.15}$$

Maximizing the posterior distribution of Λ in (2.14) with respect to λ, we can obtain the same generalized MLE for failure rate λ in (2.10). Therefore, under the truncated gamma prior distribution on Λ, the generalized MLE $\tilde{R}_{TG}(t_0)$ of the system reliability $R(t_0)$ is the same as the generalized MLE $\tilde{R}_G(t_0)$.

Tables 1 through 3 show the simulated values for the MSE of the proposed reliability estimators for the two-unit hot standby system with perfect switch when $n = 25$ and various values of λ and specified mission time t_0. From the tables, the Bayes estimator for the system reliability with respect to a truncated gamma prior distribution on the failure rate performs better than the Bayes estimators with respect to the uniform and gamma prior distributions. Under the uniform and gamma prior distributions on the failure rate the generalized MLE's for the system reliability are more efficient than the Bayes estimators for large value of the system reliability $R(t_0)$. But as value of system reliability $R(t_0)$ decreases, the Bayes estimator for the system reliability is more efficient than the generalized MLE. Under the truncated gamma prior the Bayes estimator for the system reliability performs better than the generalized MLE.

Table 1. The simulated MSE's for Bayes estimator and generalized MLE for the two-unit hot standby system reliability under the UNIF(0,λ_1) prior.

		t_0	0.5	1.0	2.0	3.0	4.0
UNIF(0,2)	$\lambda = 0.5$	$R(t_0)$	0.97350	0.90979	0.73575	0.55782	0.40600
		$\hat{R}_U(t_0)$	0.00709	0.04452	0.12583	0.12950	0.09216
		$\tilde{R}_U(t_0)$	0.00648	0.04171	0.12297	0.13066	0.09502
	$\lambda = 1.0$	$R(t_0)$	0.90979	0.73575	0.40600	0.19914	0.09157
		$\hat{R}_U(t_0)$	0.00146	0.00697	0.01170	0.00787	0.00404
		$\tilde{R}_U(t_0)$	0.00158	0.00777	0.01360	0.00899	0.00428
	$\lambda = 2.0$	$R(t_0)$	0.73575	0.40600	0.09157	0.01735	0.00301
		$\hat{R}_U(t_0)$	0.02748	0.10196	0.10542	0.04448	0.01491
		$\tilde{R}_U(t_0)$	0.02955	0.10960	0.11186	0.04558	0.01441
UNIF(0,3)	$\lambda = 0.5$	$\hat{R}_U(t_0)$	0.00779	0.04709	0.12834	0.13046	0.09243
		$\tilde{R}_U(t_0)$	0.00648	0.04171	0.12297	0.13066	0.09502
	$\lambda = 1.0$	$\hat{R}_U(t_0)$	0.00191	0.00837	0.01269	0.00812	0.00409
		$\tilde{R}_U(t_0)$	0.00158	0.00777	0.01360	0.00899	0.00428
	$\lambda = 2.0$	$\hat{R}_U(t_0)$	0.02179	0.10113	0.10495	0.04438	0.01490
		$\tilde{R}_U(t_0)$	0.02955	0.10960	0.11186	0.04558	0.01441
UNIF(0,4)	$\lambda = 0.5$	$\hat{R}_U(t_0)$	0.00781	0.04712	0.12836	0.13046	0.09244
		$\tilde{R}_U(t_0)$	0.00648	0.04171	0.12297	0.13066	0.09502
	$\lambda = 1.0$	$\hat{R}_U(t_0)$	0.00191	0.00839	0.01269	0.00813	0.00409
		$\tilde{R}_U(t_0)$	0.00158	0.00777	0.01360	0.00899	0.00428
	$\lambda = 2.0$	$\hat{R}_U(t_0)$	0.02719	0.10113	0.10495	0.04438	0.01490
		$\tilde{R}_U(t_0)$	0.02955	0.10960	0.11186	0.04558	0.01441

Table 2. The simulated MSE's for Bayes estimator and generalized MLE for the two-unit hot standby system reliability under the GAM(α,β) prior.

		t_0	0.5	1.0	2.0	3.0	4.0
GAM(1,2)	$\lambda = 0.5$	$\hat{R}_G(t_0)$	0.00681	0.04267	0.12163	0.12648	0.09069
		$\tilde{R}_G(t_0)$	0.00561	0.03743	0.11594	0.12650	0.09328
	$\lambda = 1.0$	$\hat{R}_G(t_0)$	0.00139	0.00644	0.01081	0.00751	0.00399
		$\tilde{R}_G(t_0)$	0.00115	0.00601	0.01167	0.00834	0.00416
	$\lambda = 2.0$	$\hat{R}_G(t_0)$	0.02796	0.10392	0.10769	0.04547	0.01525
		$\tilde{R}_G(t_0)$	0.03036	0.11260	0.11479	0.04670	0.01475
GAM(1,3)	$\lambda = 0.5$	$\hat{R}_G(t_0)$	0.00704	0.04382	0.12400	0.12833	0.09172
		$\tilde{R}_G(t_0)$	0.00580	0.03851	0.11835	0.12844	0.09435
	$\lambda = 1.0$	$\hat{R}_G(t_0)$	0.00146	0.00669	0.01096	0.00745	0.00389
		$\tilde{R}_G(t_0)$	0.00120	0.00619	0.01177	0.00826	0.00407
	$\lambda = 2.0$	$\hat{R}_G(t_0)$	0.02762	0.10246	0.10573	0.04444	0.01484
		$\tilde{R}_G(t_0)$	0.03002	0.11110	0.11269	0.04560	0.01433
GAM(1,4)	$\lambda = 0.5$	$\hat{R}_G(t_0)$	0.00715	0.04441	0.12520	0.12926	0.09224
		$\tilde{R}_G(t_0)$	0.00590	0.03906	0.11957	0.12942	0.09489
	$\lambda = 1.0$	$\hat{R}_G(t_0)$	0.00150	0.00683	0.01105	0.00742	0.00384
		$\tilde{R}_G(t_0)$	0.00123	0.00629	0.01183	0.00823	0.00403
	$\lambda = 2.0$	$\hat{R}_G(t_0)$	0.02745	0.10174	0.10475	0.04392	0.01463
		$\tilde{R}_G(t_0)$	0.02985	0.11035	0.11165	0.04506	0.01412

Table 3. The simulated MSE's for Bayes estimator and generalized MLE for the two-unit hot standby system reliability under the TGAM(α,β) prior.

		t_0	0.5	1.0	2.0	3.0	4.0
TGAM(1,2)	$\lambda = 0.$	$\hat{R}_{TG}(t_0)$	0.00495	0.03389	0.10716	0.11731	0.08642
		$\tilde{R}_{TG}(t_0)$	0.00561	0.03743	0.11594	0.12650	0.09328
	$\lambda = 1.0$	$\hat{R}_{TG}(t_0)$	0.00062	0.00358	0.00830	0.00699	0.00409
		$\tilde{R}_{TG}(t_0)$	0.00115	0.00601	0.01167	0.00834	0.00416
	$\lambda = 2.$	$\hat{R}_{TG}(t_0)$	0.03016	0.11227	0.11659	0.04933	0.01659
		$\tilde{R}_{TG}(t_0)$	0.03036	0.11260	0.11479	0.04670	0.01475
TGAM(1,3)	$\lambda = 0.$	$\hat{R}_{TG}(t_0)$	0.00505	0.03453	0.10880	0.11879	0.08732
		$\tilde{R}_{TG}(t_0)$	0.00580	0.03851	0.11835	0.12844	0.09435
	$\lambda = 1.0$	$\hat{R}_{TG}(t_0)$	0.00063	0.00360	0.00821	0.00683	0.00396
		$\tilde{R}_{TG}(t_0)$	0.00120	0.00619	0.01177	0.00826	0.00407
	$\lambda = 2.$	$\hat{R}_{TG}(t_0)$	0.02991	0.11111	0.11484	0.04833	0.01618
		$\tilde{R}_{TG}(t_0)$	0.03002	0.11110	0.11269	0.04560	0.01433
TGAM(1,4)	$\lambda = 0.$	$\hat{R}_{TG}(t_0)$	0.00051	0.03485	0.10963	0.11954	0.08777
		$\tilde{R}_{TG}(t_0)$	0.00590	0.03906	0.11957	0.12942	0.09489
	$\lambda = 1.0$	$\hat{R}_{TG}(t_0)$	0.00063	0.00361	0.00817	0.00675	0.00390
		$\tilde{R}_{TG}(t_0)$	0.00123	0.00629	0.01183	0.00823	0.00403
	$\lambda = 2.$	$\hat{R}_{TG}(t_0)$	0.02978	0.11053	0.11397	0.04784	0.01597
		$\tilde{R}_{TG}(t_0)$	0.02985	0.11035	0.11165	0.04506	0.01412

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