A TREND IN THE INTERPOLATION AND APPROXIMATION THEORY

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1. Interpolation by Radial Basis functions

For a given set X, we want to interpolate a function $g : X \to \mathbb{R}$. For prescribed nodes $x_1, \dots, x_n$, if a function F has the property $F(x_i) = g(x_i)$ for $1 \le i \le n$, then we say that F interpolates g at the nodes. We usually choose the function F to be well-behaved and easily-computed function in an s-dimensional linear space.

Problem: Suppose that $x_1, \dots, x_n \in \mathbb{R}^s$ and that data $d_1, \dots, d_n$ are prescribed. Does there exist a unique set of scalars $a_1, \dots, a_n$ such that

$$\sum_{j=1}^{n} a_j f(\|x_i - x_j\|^2) = d_i, \qquad i = 1, \dots, n$$

i. e. for what functions f the interpolation matrix

$$A_{ij} = f(\|x_i - x_j\|^2)$$

is nonsingular?

DEFINITION 1 *A function $f : [0, \infty) \to [0, \infty)$ is said to be* completely monotone *if*

(a) $f \in C[0, \infty)$ and $f \in C^\infty(0, \infty)$

(b) $(-1)^k f^{(k)}(t) \ge 0$ for $t > 0$ and $k = 0, 1, 2, \dots$

Let CM denote the class of completely monotone functions.

Examples. Typical examples of completely monotone functions that can be verified directly from the definition are

1. $f(t) = a \qquad (a \ge 0)$

2. $f(t) = (t+a)^b \qquad (a > 0 > b)$

3. $f(t) = e^{-at} \qquad (a > 0)$

THEOREM 1 *(The Schoenberg Interpolation Theorem [S1])*
If f is completely monotone but not constant on $[0,\infty)$, then for any n and for any distinct points $x_1, x_2, \ldots, x_n$ in any real inner product space the $n \times n$ matrix A defined by $A_{ij} = f(\|x_i - x_j\|^2)$ is positive definite (and therefore nonsingular).

Examples. There are other useful functions, *not* included in Schoenberg's Theorem, that lead to nonsingular interpolation problems.

(4) $f(t) = \sqrt{t}$

(5) $f(t) = \sqrt{1+t}$

(6) $f(t) = \log(1+t)$

The function in (5) is called a "multiquadric"; it has been found useful in geophysics (see [Ha], [Ha2]). Examples (4)–(6) can be related to a Theorem of Micchelli [Mi] Following Sun [Su1] we define the appropriate function class as follows:

DEFINITION 2 *Let DM denote the class of all functions $f : [0,\infty) \to [0,\infty)$ such that*

(a) $f \in C[0,\infty)$ and $f \in C^\infty(0,\infty)$

(b) f' is completely monotone but not constant on $(0,\infty)$.

THEOREM 2 *(The Micchelli Interpolation Theorem [Mi]) If f belongs to DM then for any n distinct points $x_1, x_2, \ldots, x_n$ in a real inner product space the matrix $A_{ij} = f(\|x_i - x_j\|^2)$ is nonsingular.*

If $f(\|\cdot\|)$ is a positive definite function, then the interpolation matrix $A_{ij} = f(\|x_i - x_j\|^2)$ will be nonnegative definite which maybe singular. Thus we need to consider a strictly positive definite function. A sufficient condition for strictly positive definiteness is given in the following theorem.

THEOREM 3 *[Che] Let μ be a nonnegative finite-valued Borel measure on $\mathbb{R}^s$. If the carrier of μ is not a set of Lebesgue measure 0, then the Fourier transform of μ is a strictly positive definite function on $\mathbb{R}^s$.*

Open Problem: Is there a Bochner-type theorem for *strictly* positive definite function?

2. Approximation by Radial Basis functions

In the previous section we considered the problem of interpolation on $\mathbb{R}^s$ by the functions of the form

$$F(x) = \sum_{j=1}^{n} c_j f(\|x - x_j\|^2)$$

Now we consider the question of approximating a given function by F.

Problem: For what functions f is it true that for each compact set Q in $\mathbb{R}^s$ the set

$$\{x \mapsto f(\|x - y\|^2) : y \in Q\}$$

is fundamental in $C(Q)$?

DEFINITION 3 *A set V in a normed linear space E is said to be* fundamental *if*

$$closure(span(V)) = E.$$

In other words, the set of all linear combinations of elements of V is dense in E.

remark The above definition can be found in Banach's book, [Ban]. The most famous example of a fundamental set in approximation theory is the set of monomials $\{1, x, x^2, \ldots\}$ in $C[0,1]$. (For other classical examples of fundamental sets, see [C1]).

LEMMA 1 *For a subset V in a normed linear space E, the following two properties are equivalent:*
(a) V is fundamental (i. e., its linear span is dense in E).
(b) V is total (i. e. $V^{\perp} = 0$, or 0 is the only element of E^* that annihilates V).

THEOREM 4 *[Li] Let f be completely monotone but not constant on $[0,\infty)$. Let Q be a compact subset (containing at least 2 points) in $\mathbb{R}^s$. Then the set of functions*

$$\{x \mapsto f(\|x - y\|_2^2) : y \in Q\}$$

is fundamental in $C(Q)$.

THEOREM 5 *[Che] Let $f \in C[0,\infty) \cap C^\infty(0,\infty)$, and assume that f' is completely monotone but not constant on $[0,\infty)$. Assume that $f'(\infty) = 0$. Then for any compact set Q in $\mathbb{R}^s$ the set of functions*

$$\{x \mapsto f(\|x-y\|_2^2) : y \in Q\}$$

is fundamental in $C(Q)$.

THEOREM 6 *[Yun] Let $f \in C(\mathbb{R}^s \times \mathbb{R}^s)$ and the double integral*

$$\int_{\mathbb{R}^s}\int_{\mathbb{R}^s} f(\|x-y\|^2)d\mu(x)d\mu(y)$$

is positive for any compactly supported nontrivial signed Borel measure μ on $\mathbb{R}^s$. Then for any compact set Q in $\mathbb{R}^s$ the set of functions

$$\{x \mapsto f(\|x-y\|^2) : y \in Q\}$$

is fundamental in $C(Q)$.

3. Ridge Functions

DEFINITION 4 *Let X be a normed linear space. A function $f : X \to \mathbb{R}$ is called a* ridge function *if it can be represented in the form $f = g \circ \phi$, where $g \in C(\mathbb{R})$ and $\phi \in X^*$.*

Example. Every continuous linear functional on $\mathbb{R}^s$ is the form

$$\phi(x) = \alpha_1\xi_1 + \cdots + \alpha_s\xi_s =< a, x >$$

where $x = (\xi_1, \cdots, \xi_s)$ and $a = (\alpha_1, \cdots, \alpha_s)$. Thus a ridge function on $\mathbb{R}^s$ is the form

$$f(x) = g(\alpha_1\xi_1 + \cdots + \alpha_s\xi_s) = g(< a, x >)$$

Since the graph of a ridge function is a ruled surface, a single ridge function is not good enough to approximate an arbitrary continuous function on X. Thus we consider linear combinations of ridge functions.

$$f = \sum_{i=1}^{m} c_i g_i \circ \phi_i \quad (g_i \in C(\mathbb{R}), \phi_i \in X^*)$$

Not all continuous functions on X are linear combinations of ridge functions. But every continuous function on X can be well approximated by such linear combinations, as we shall see below.

For the space $C(X)$ of all continuous real-valued functions on the normed linear space X, we use the topology of uniform convergence on compacta.

A subset of $C(X)$ is said to be fundamental if its linear span is dense. For example, the monomial functions $\{1, x, x^2, \cdots\}$ is fundamental in $C(\mathbb{R})$ by the following version of the Stone-Weierstrass theorem.

THEOREM 7 *(Stone-Weierstrass Theorem) Let X be a normed linear space. If $\mathcal{A}$ is a subalgebra of $C(X)$ that contains constants and seperates the points of X, then $\mathcal{A}$ is dense in $C(X)$.*

Now we state a basic problem in the study of ridge functions.

Problem: For a given normed linear space X, a given subset $G \in C(\mathbb{R})$, and a given subset $\Phi \in X^*$, determine whether the set

$$\{g \circ \phi : g \in G, \phi \in \Phi\}$$

is fundamental in $C(X)$.

Next we quote some basic results concerning this problem from [Su1], [C2].

THEOREM 8 *Let X be a normed linear space. The set of ridge functions $\{exp \circ \phi : \phi \in X^*\}$ is fundamental in $C(X)$.*

THEOREM 9 *The set of ridge functions*

$$\{p \circ \phi : \phi \in \Pi(\mathbb{R}), \phi \in X^*\}$$

is fundamental in $C(X)$, for any normed linear space.

THEOREM 10 *Let g be a continuous function on $\mathbb{R}$ such that the limits of $g(t)$ as $t \to \infty$ and $t \to -\infty$ exist and are different. Put $g_{i,j}(t) = g(jt + i)$. Then*

$$\{g_{i,j} : i, j \in \mathbf{Z}\}$$

is fundamental in $C(\mathbb{R})$.

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