FREE ACTIONS ON THE NILMANIFOLD

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ABSTRACT. We classify free actions of finite abelian groups on the
3-dimensional nilmanifold, up to topological conjugacy. By the works
of Bieberbach and Waldhausen, this classification problem is reduced
to classifying all normal subgroups of almost Bieberbach groups of
finite index, up to affine conjugacy.

1. Introduction

The general question of classifying finite group actions on a closed
3-manifold is very hard. However, the actions on a 3-dimensional
torus can be understood easily by the works of Bieberbach and Wald-
hausen([5, 9]). We shall study only free actions of finite abelian groups
G on the 3-dimensional nilmanifold.

Let L be a simply connected, nilpotent Lie group. Then $\mathrm{Aff}(L) =
L \rtimes \mathrm{Aut}(L)$ is called the affine group of L, where the group operation is
given by $(g, \alpha)(h, \beta) = (g \cdot \alpha(h), \alpha\beta)$ and $\mathrm{Aff}(L)$ acts on L by $(g, \alpha)z =
g \cdot \alpha(z)$ for $(g, \alpha), (h, \beta) \in \mathrm{Aff}(L)$ and $z \in L$. Let K be any maximal
compact subgroup of $\mathrm{Aut}(L)$. Then a discrete uniform subgroup E
of $L \rtimes K$ is called an *almost crystallographic group*. When E is a
torsion-free, it is called an *almost Bieberbach group* and the coset
space $E \backslash L$ is an infra-nilmanifold. (In case $E \subset L$, $E \backslash L$ is called
a nilmanifold.) If L is abelian ($\cong \mathbb{R}^n$ for some n), this terminology

reduces to a *crystallographic group*, a *Bieberbach group* and a *flat Riemannian manifold* respectively.

The maximal compact subgroup K can be chosen so that $L \rtimes K$ equals $\text{Isom}(L)$. Consequently, a closed 3–dimensional manifold has a Nil–geometry if and only if it is an infra–nilmanifold. It is well known that infra-nilmanifolds are determined completely (up to affine diffeomorphism) by their fundamental group E.

Let G be a finite group acting freely on a nilmanifold $\mathcal{N}$. Then clearly, $M = \mathcal{N}/G$ is a topological manifold, and $\Gamma = \pi_1(\mathcal{N}/G)$ is an abstract Bieberbach group. Let N be the subgroup of Γ corresponding to $\pi_1(\mathcal{N})$. Let Γ' be an embedding of Γ into $\text{Aff}(L)$ as a cocompact subgroup, and let N' be the image of N. Then the quotient group $G' = \Gamma'/N'$ acts freely on the nilmanifold $\mathcal{N}' = L/N'$. Moreover, $M' = \mathcal{N}'/G'$ is an infra-nilmanifold. Thus, a finite free topological action $(G, \mathcal{N})$ gives rise to an isometric action $(G', \mathcal{N}')$ on a nilmanifold. Clearly, $\mathcal{N}/G$ and $\mathcal{N}'/G'$ are sufficiently large, see [4, Proposition 2]. By works of Waldhausen and Heil ([3, Theorem A]), M is homeomorphic to M'.

DEFINITION 1.1. Let groups G_i act on manifolds M_i, for $i = 1, 2$. The action (G_1, M_1) is *topologically conjugate* to (G_2, M_2) if there exists an isomorphism $\theta \colon G_1 \to G_2$ and a homeomorphism $h \colon M_1 \to M_2$ such that $h(g \cdot x) = \theta(g) \cdot h(x)$ for all $x \in M_1$ and all $g \in G_1$. When $G_1 = G_2$ and $M_1 = M_2$, topologically conjugate is the same as *weakly equivariant*.

For $\mathcal{N}/G$ and $\mathcal{N}'/G'$ being homeomorphic implies that the two actions $(G, \mathcal{N})$ and $(G', \mathcal{N}')$ are topologically conjugate. Consequently, a free finite action $(G, \mathcal{N})$ gives rise to a topologically conjugate isometric action $(G', \mathcal{N}')$ on a nilmanifold $\mathcal{N}'$. Such a pair $(G', \mathcal{N}')$ is not unique. However, by the following theorem which has been obtained

by Lee and Raymond ([7]), all the others are topologically conjugate.

THEOREM 1.2. *Let $\phi : E \to E'$ be an isomorphism between two almost crystallographic groups of L. Then ϕ is conjugation by an element of $\mathrm{Aff}(L)$.*

Consequenly, to classify all free actions by finite groups on a nilmanifold, it is enough to classify only free isometric actions by finite groups on a nilmanifold. Lee proved the following theorem([6]).

THEOREM 1.3. *Let $1 \to N \to E \to F \to 1$ be any extension of a finitely generated, torsion-free nilpotent group N by a finite group F. Then there exists a finite characteristic subgroup Z of E such that E/Z is an almost crystallographic group. In fact, Z is the subgroup consisting of all torsion elements of $C_E(N)$, the centralizer of N in E.*

Therefore, a finitely generated group E is an almost crystallographic group if and only if it has a torsion-free maximal normal nilpotent subgroup of finite index.

2. Criteria for conjugacy

In this section, we develop a technique for finding and classifying all possible finite group actions on the 3-dimensional nilmanifold. The problem will be reduced to a purely group-theoretic one.

Consider the 3-dimensional Heisenberg group

$$\mathcal{H} = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} : x, y, z \in \mathbb{R} \right\}$$

which is connected, simply connected and two-step nilpotent. Then the center $\mathcal{Z}(\mathcal{H})$ of $\mathcal{H}$ is 1-dimensional, consisting of all matrices with $x = y = 0$. The quotient $\mathcal{H}/\mathcal{Z}(\mathcal{H})$ is isomorphic to $\mathbb{R}^2$ so that

$$1 \to \mathbb{R} \to \mathcal{H} \to \mathbb{R}^2 \to 1$$

is exact. The automorphism group is

$$\text{Aut}(\mathcal{H}) = \mathbb{R}^2 \rtimes \text{GL}(2, \mathbb{R}),$$

where the factor $\text{GL}(2, \mathbb{R})$ comes from the automorphisms of the quotient $\mathbb{R}^2$, and the $\mathbb{R}^2$ factor is the inner automorphisms. In fact, an element

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{GL}(2, \mathbb{R})$$

yields an automorphism of $\mathcal{H}$ mapping

$$\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & ax + by & z' \\ 0 & 1 & cx + dy \\ 0 & 0 & 1 \end{bmatrix}$$

where

$$z' = (ad - bc)z + \frac{1}{2}(acx^2 + 2bcxy + bdy^2).$$

An element

$$U = \begin{bmatrix} u & v \end{bmatrix} \in \mathbb{R}^2 \subset \text{Aut}(\mathcal{H})$$

represents an automorphism of $\mathcal{H}$ mapping

$$\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & x & z - vx + uy \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}.$$

Throughout this paper, we shall denote the Heisenberg group simply by $\mathcal{H}$.

Let Γ be any lattice of $\mathcal{H}$. Then $\mathbb{Z} = \Gamma \cap \mathcal{Z}(\mathcal{H})$ and $\Gamma / \Gamma \cap \mathcal{Z}(\mathcal{H})$ are lattices of $\mathcal{Z}(\mathcal{H})$ and $\mathcal{H}/\mathcal{Z}(\mathcal{H})$, respectively. Therefore, the lattice Γ is an extension of $\mathbb{Z}$ by $\mathbb{Z}^2$, that is,

$$1 \to \mathbb{Z} \to \Gamma \to \mathbb{Z}^2 \to 1$$

We take e_1, e_2 and e_3 in Γ for some $k(\neq 0) \in \mathbb{Z}$, where

$$
e_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 1 & 0 & 1/k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.
$$

Thus

$$
\Gamma = \langle e_1, e_2, e_3 \mid [e_1, e_3] = [e_2, e_3] = 1, \ [e_1, e_2] = e_3^k \rangle.
$$

It is easy to see that every lattice of $\mathcal{H}$ is isomorphic to Γ for some $k > 0$. From now on, we denote the lattice determined as above by Γ_k. A group is *virtually p-step nilpotent* if it contains a normal subgroup which is p-step nilpotent and of finite index. Proposition 2.1 gives a characterization of an almost Bieberbach group.

PROPOSITION 2.1. *An abstract group π is the fundamental group of a 3-dimensional infra-nilmanifold if and only if π is a torsion-free, virtually nilpotent group of rank 3. Consequently, such a π contains Γ_k for some $k > 0$ as a normal subgroup of finite index.*

Proof. Suppose that M is a 3-dimensional infra-nilmanifold Then M is finitely covered by a nilmanifold $\Gamma \backslash \mathcal{H}$, where Γ is a lattice of $\mathcal{H}$. Then necessarily, $\Gamma = \Gamma_k$ for some $k > 0$, and hence,

$$
1 \rightarrow \Gamma_k \rightarrow \pi_1(M) \rightarrow \Phi \rightarrow 1
$$

is an extension of Γ_k by a finite group Φ. Thus $\pi_1(M)$ contains Γ_k as a nilpotent subgroup of rank 3. Indeed, Γ_k is the unique maximal normal nilpotent subgroup of $\pi_1(M)$.

Conversely, π fits an exact sequence $1 \rightarrow N \rightarrow \pi \rightarrow F \rightarrow 1$ where N is a finitely generated, torsion-free nilpotent group and F is a finite group. Then clearly $N = \Gamma_k$ for some k. Thus Theorem 1.3 completes the proof. $\qquad\square$

In fact, the fundamental group of an infra-nilmanifold determines the space completely as in the case of a compact flat Riemannian manifold.

THEOREM 2.2. *Let M_1, M_2 be infra-nilmanifolds. Then the following conditions are equivalent.*

    (1) *M_1 is affinely diffeomorphic to M_2.*

    (2) *M_1 is homeomorphic to M_2.*

    (3) *The fundamental group $\pi_1(M_1)$ is isomorphic to $\pi_1(M_2)$.*

Proof. The statement (3) is equivalent to M_1 being homotopy equivalent to M_2, since both are aspherical manifolds. Now, homotopy equivalent infra-nilmanifolds are affinely diffeomorphic ([7]), so (iii) implies (i).   □

Let $M = \mathcal{H}/\pi$ be a 3-dimensional infra-nilmanifold where π is a subgroup of $\mathrm{Aff}(\mathcal{H}) = \mathcal{H} \rtimes \mathrm{Aut}(\mathcal{H})$. Then there is a diffeomorphism f between $\mathcal{H}$ and $\mathbb{R}^3$, and an isomorphism φ between π and π' where π' is a subgroup of $\mathrm{Aff}(3) = \mathbb{R}^3 \rtimes \mathrm{GL}(3,\mathbb{R})$ such that $(\pi, \mathcal{H})$ and $(\pi', \mathbb{R}^3)$ are weakly equivariant. Therefore, an infra-nilmanifold $M = \mathcal{H}/\pi$ is diffeomorphic to an affine manifold $M' = \mathbb{R}^3/\pi'$.

DEFINITION 2.3. Let $\Gamma \subset \mathrm{Aff}(\mathcal{H})$ be an almost Bieberbach group, and let N_1, N_2 be subgroups of Γ. We say that (N_1, Γ) is *affinely conjugate* to (N_2, Γ) if there exists an element $(t, T) \in \mathrm{Aff}(\mathcal{H})$ such that $(t, T)\Gamma(t, T)^{-1} = \Gamma$ and $(t, T)N_1(t, T)^{-1} = N_2$.

Let $(G, \mathcal{N})$ be a free affine action of a finite abelian group G on a nilmanifold $\mathcal{N}$. Then $\mathcal{N}/G$ is an infra-nilmanifold. Let $\Gamma = \pi_1(\mathcal{N}/G)$, and $N = \pi_1(\mathcal{N})$. Then Γ is an almost Bieberbach group. In fact, Since the covering projection $\mathcal{N} \to \mathcal{N}/G$ is regular, N is a normal subgroup of Γ. Since the pure translations in Γ, $\mathcal{Z} = \Gamma \cap \mathcal{H}$, is the

unique maximal normal subgroup of Γ, the normal subgroup N must be in $\mathcal{Z}$.

Our classification problem of free abelian group actions $(G, \mathcal{N})$ with $\pi_1(\mathcal{N}/G) \cong \Gamma$ can be solved by two steps:

(I) Find all normal free subgroups N of Γ of finite index and classify (N, Γ) up to affine conjugacy.

(II) Realize the finite abelian group $G = \Gamma/N$ as an action on the nilmanifold $\mathcal{H}/N$.

For (I), we need the following. Let $\Gamma \subset \mathrm{Aff}(\mathcal{H}) = \mathcal{H} \rtimes \mathrm{Aut}(\mathcal{H})$ be an almost Bieberbach group; and let

$$t_1 = (e_1, I), \qquad t_2 = (e_2, I), \qquad t_3 = (e_3, I),$$

where I is the identity matrix in $\mathrm{Aut}(\mathcal{H}) = \mathbb{R}^2 \rtimes \mathrm{GL}(2, \mathbb{R})$. Then $\mathcal{Z} = \langle t_1, t_2, t_3 \rangle$ is the maximal normal free subgroup of Γ. Let N be a subgroup of $\mathcal{Z}$ of rank 3 which is normal in Γ, and let

$$\mathcal{B} = \{ t_1^{\ell_1} t_2^{m_1} t_3^{n_1}, \; t_1^{\ell_2} t_2^{m_2} t_3^{n_2}, t_1^{\ell_3} t_2^{m_3} t_3^{n_3} \}$$

be an ordered set of generators for N. More precisely,

$$t_1^{\ell_i} t_2^{m_i} t_3^{n_i} = \left(e_1^{\ell_i} e_2^{m_i} e_3^{n_i}, I \right), \quad i = 1, 2, 3.$$

We shall represent the particular ordered basis $\mathcal{B}$ of N as following:

$$[\mathcal{B}] = \left(e_1^{\ell_1} e_2^{m_1} e_3^{n_1}, e_1^{\ell_2} e_2^{m_2} e_3^{n_2}, e_1^{\ell_3} e_2^{m_3} e_3^{n_3} \right)$$

$$\Longleftrightarrow \quad \begin{matrix} \langle t_1^{\ell_1} t_2^{m_1} t_3^{n_1}, \; t_1^{\ell_2} t_2^{m_2} t_3^{n_2}, \; t_1^{\ell_3} t_2^{m_3} t_3^{n_3} \rangle \\ \text{ordered basis of a subgroup} \\ N \text{ of } \mathcal{Z} \end{matrix} \quad .$$

Let $d_1 = (\ell_1, \ell_2, \ell_3)$. Then there exist integers p, q and r such that $p\ell_1 + q\ell_2 + r\ell_3 = d_1$. Thus we have

$$(e_1^{\ell_1} e_2^{m_1} e_3^{n_1})^p (e_1^{\ell_2} e_2^{m_2} e_3^{n_2})^q (e_1^{\ell_3} e_2^{m_3} e_3^{n_3})^r = e_1^{d_1} e_2^{m'} e_3^{n'}.$$

Now using $e_1^{d_1} e_2^{m'} e_3^{n'}$, we can kill off all e_1 powers from all generators and get a new set $\{e_2^{m'_1} e_3^{n'_1}, e_2^{m'_2} e_3^{n'_2}, e_2^{m'_3} e_3^{n'_3}\}$. Let $d_2 = (m'_1, m'_2, m'_3)$. Then using a similar method repeatedly as above, we can change $[\mathcal{B}]$ to a new generating set $[\mathcal{B}'] = \left(e_1^{d_1} e_2^{m} e_3^{n_1}, e_2^{d_2} e_3^{n_2}, e_3^{d_3} \right)$, where $d_2 > m$, $d_3 > n_i$, $i = 1, 2$. Thus we have the following Lemma.

LEMMA 2.4. *Any free normal subgroup N of Γ has an ordered set of generators of the form*

$$\langle t_1^{d_1} t_2^{m} t_3^{n_1}, \ t_2^{d_2} t_3^{n_2}, \ t_3^{d_3} \rangle.$$

Let us denote the normalizer of Γ by $N_{\mathrm{Aff}(\mathcal{H})}(\Gamma)$. The maximal normal free subgroup $\mathcal{Z}$ of Γ is characteristic (i.e., invariant under any automorphism of Γ). Under our representation of Γ into $\mathrm{Aff}(\mathcal{H})$, the subgroup Γ is a lattice of $\mathcal{H}$. Therefore, matrix parts of elements of $N_{\mathrm{Aff}(\mathcal{H})}(\Gamma)$ are integral.

To make the exposition easier, we introduce some more notations. Let N_1, N_2 be free normal subgroups of Γ; $\mathcal{B}_1$, $\mathcal{B}_2$ be bases for N_1, N_2, respectively. If there is $Y \in \mathrm{GL}(2, \mathbb{Z}) \subset \mathrm{Aut}(\mathcal{H})$ such that $[\mathcal{B}_1]Y = [\mathcal{B}_2]$, then we say $[\mathcal{B}_1] \underset{C}{\sim} [\mathcal{B}_2]$. Similarly, if there exists $(t, T) \in N_{\mathrm{Aff}(\mathcal{H})}(\Gamma)$ so that $[\mathcal{B}_2] = t \cdot T([\mathcal{B}_1]) \cdot t^{-1}$, then we say $[\mathcal{B}_1] \underset{R}{\sim} [\mathcal{B}_2]$. Note that $\underset{C}{\sim}$ is the right action of $\mathrm{Aut}(N)$ on a fixed basis so that it does not change Γ and its normal subgroup. It is an operation that picks a new set of generators. Therefore, if $[\mathcal{B}_1] \underset{C}{\sim} [\mathcal{B}_2]$, then $N_1 = N_2$. On the other hand, $\underset{R}{\sim}$ is the row operation on the matrix leaving Γ invariant. If (t, T) is in the normalizer of Γ, then it gives a new representation of Γ. Moreover, even if $[\mathcal{B}_1] \underset{R}{\sim} [\mathcal{B}_2]$, N_1 and N_2 will generally be different subgroups of Γ.

The following proposition is a working criterion for affine conjugacy. All calculations will be done by this method.

PROPOSITION 2.5. *Let N_1, N_2 be free normal subgroups of a Bieberbach group Γ. Then (N_1, Γ) is affinely conjugate to (N_2, Γ) if and only if for any ordered set of generators $\mathcal{B}_1, \mathcal{B}_2$ for N_1, N_2, respectively, there exist $(t, T) \in N_{\mathrm{Aff}(\mathcal{H})}(\Gamma)$ and $Y \in GL(2, \mathbb{Z})$ such that*

$$[\mathcal{B}_2] Y = t \cdot T([\mathcal{B}_1]) \cdot t^{-1}.$$

Proof. Let $\mathcal{B}_1$, $\mathcal{B}_2$ be any ordered bases for N_1, N_2, respectively, say

$$\mathcal{B}_1 = \{(a_1, I), (a_2, I), (a_3, I)\}, \quad \mathcal{B}_2 = \{(b_1, I), (b_2, I), (b_3, I)\}.$$

Then $[\mathcal{B}_1] = [a_1, a_2, a_3]$ and $[\mathcal{B}_2] = [b_1, b_2, b_3]$. Suppose there exists $(t, T) \in \mathrm{Aff}(\mathcal{H})$ giving rise to an affine conjugacy from (N_1, Γ) to (N_2, Γ). So the conjugation by (t, T) maps Γ to Γ itself and N_1 to N_2. Since $(t, T)(a_i, I)(t, T)^{-1} = (t \cdot T(a_i) \cdot t^{-1}, I)$, for $i = 1, 2, 3$, clearly $\{t \cdot T(a_1) \cdot t^{-1}, t \cdot T(a_2) \cdot t^{-1}, t \cdot T(a_3) \cdot t^{-1}\}$ is a new ordered set of generators for N_2. This is related to the original ordered set of generators $\mathcal{B}_2$ by an integral matrix $Y \in GL(2, \mathbb{Z})$. Thus

$$t \cdot T([\mathcal{B}_1]) \cdot t^{-1} = [t \cdot T(a_1) \cdot t^{-1}, t \cdot T(a_2) \cdot t^{-1}, t \cdot T(a_3) \cdot t^{-1}] = [\mathcal{B}_2] Y.$$

The converse is easy. $\qquad \square$

For convenience, in the rest of the paper we shall use the notation $N_1 \sim N_2$ if $[\mathcal{B}_2] Y = t \cdot T([\mathcal{B}_1]) \cdot t^{-1}$ as in Proposition 2.5.

Now, the first step (I) is a purely group-theoretic problem and can be handled by Proposition 2.5. Firstly, we need to calculate the normalizer $N_{\mathrm{Aff}(\mathcal{H})}(\Gamma)$. Let $(t, T) \in \mathrm{Aff}(\mathcal{H})$. For (t, T) to normailize the maximal free subgroup $\mathcal{Z}$ of Γ, it is necessary and sufficient that the matrix T to be in $GL(2, \mathbb{Z})$. To take care of the rest, Pick a finite

subset F of Γ whose image in the quotient (=holonomy) group $\Gamma/\mathcal{Z}$ is a set of generators. Find all $(t, T) \in \mathcal{H} \rtimes \mathrm{GL}(2, \mathbb{Z})$ such that

$$(t, T)(a, A)(t, T)^{-1} \in \Gamma$$

for all $(a, A) \in F$. It is equivalent to the following condition

$$(t, T)(a, A)(t, T)^{-1}(a, A)^{-1} \in \Gamma \cap \mathcal{H} = \mathcal{Z} \iff t \cdot A(t^{-1}) \in \mathcal{Z}.$$

In dimension 3, F can be taken so that it has cardinality at most 2. For example, $\Gamma = \langle t_1, t_2, t_3, \alpha \mid [t_1, t_2] = t_3^{2k}, \alpha = (a, A), \alpha^2 = t_3 \rangle$, where

$$a = \begin{bmatrix} 1 & 0 & \frac{1}{4k} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

One can take $F = \{\alpha = (a, A)\}$ a singleton. Thus we obtain

$$N_{\mathrm{Aff}(\mathcal{H})}(\Gamma) = \left\{ t = \begin{bmatrix} 1 & \frac{p}{2} & * \\ 0 & 1 & \frac{q}{2} \\ 0 & 0 & 1 \end{bmatrix}, \ T \in \mathrm{GL}(2, \mathbb{Z}), \ pq \in 2\mathbb{Z} \right\}.$$

The second part (II) "Realization" can be done by the following procedure. Let Γ be an almost Bieberbach group, and N be a normal subgroup of Γ with $G = \Gamma/N$ finite. To describe the natural affine action of G on the nilmanifold $\mathcal{H}/N$, we must make the nilmanifold the *standard nilmanifold*, and describe the action on the universal covering level. In other words, the action of G should be defined on $\mathcal{H}$ as affine maps (this is really explaining the liftings of a set of generators of G in Γ), and simply say that our action is the affine action modulo the standard lattice $\mathcal{Z}$. It is quite easy to achieve this. Let $\{(a_1, I), (a_2, I), (a_3, I)\}$ be a generating set for N. Form a matrix $B \in \mathrm{Aut}(\mathcal{H})$ such that $B(a_i) = e_i$, for i=1,2,3. Therefore, the conjugation by $(I, B) \in \mathrm{Aff}(\mathcal{H})$ maps Γ into another almost

Bieberbach group in such a way that the generating set for N becomes the standard basis for $\mathcal{Z}$. Suppose $\{\alpha_1, \cdots, \alpha_m\}, (m \leq 3)$ generates the quotient group G when project down via $\Gamma \to G$, then $\{(I, B)\alpha_1(I, B^{-1}), (I, B)\alpha_2(I, B^{-1}), (I, B)\alpha_3(I, B^{-1})\}$ describes the action of G on the standard nilmanifold.

3. Free actions of finite abelian groups G on the nilmanifold

It is well known that all 3-dimensional infra-nilmanifolds are Seifert manifolds. A classification of the 3-dimensional Seifert manifolds with solvable fundamental group (amongst them infra-nilmanifolds) is found in Orlik's book ([10, Theorem 1., p.142]). Assume M is a 3-dimensional infra-nilmanifold. Then M has a Seifert bundle structure; namely, M is a circle bundle over a 2-dimensional orbifold with singularities. Recently it is known ([2, Proposition 6.1.]) that there are only 15 kinds of distinct closed 3-dimensional manifolds M with a Nil-geometry up to Seifert local invariant.

In this section, we shall deal with only one out of 15 distinct almost Bieberbach groups up to Seifert local invariant. The other cases can be done similarly and will be published with K. B. Lee ([8]). From now on, let us denote Γ by

$$\Gamma = \langle t_1, t_2, t_3, \alpha \mid [t_1, t_2] = t_3^{2k}, \ \alpha^2 = t_3 \rangle,$$

where $t_1 = (e_1, I), t_2 = (e_2, I), t_3^{2k} = (e_3, I)$, and

$$\alpha = \left(\begin{bmatrix} 1 & 0 & \frac{1}{4k} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right).$$

Here I is the identity matrix in $\mathrm{Aut}(\mathcal{H}) = \mathbb{R}^2 \rtimes \mathrm{GL}(2, \mathbb{R})$. Note that $[\Gamma, \Gamma] = \langle t_1^{-2}, t_2^{-2} \rangle$, its holonomy group is $\mathbb{Z}_2$ and the fist homology group is $H_1(\Gamma; \mathbb{Z}) = \mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Recall that for integers p and q,

$$N_{\mathrm{Aff}(\mathcal{H})}(\Gamma) = \left\{ t = \begin{bmatrix} 1 & \frac{p}{2} & * \\ 0 & 1 & \frac{q}{2} \\ 0 & 0 & 1 \end{bmatrix}, \ T \in \mathrm{GL}(2, \mathbb{Z}), \ pq \in 2\mathbb{Z} \right\}.$$

We shall study free actions of finite abelian groups G on the nilmanifold $\mathcal{N}$ which yield an infra-nilmanifold homeomorphic to $\mathcal{H}/\Gamma$.

THEOREM 3.1. *The following table gives a complete list of all free actions (up to topological conjugacy) of finite abelian groups G on $\mathcal{N}$ which yield an orbit manifold homeomorphic to $\mathcal{H}/\Gamma$.*

Group G	Conjugacy classes of normal free subgroups
$\mathbb{Z}_{4k}$	$K = \langle \alpha^{4k}, t_1, t_2 \rangle$
$\mathbb{Z}_{8k} \times \mathbb{Z}_2$	$N = \langle \alpha^{8k}, t_1, t_2^2 \rangle$
$\mathbb{Z}_{16k} \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$L = \langle \alpha^{16k}, t_1^2, t_2^2 \rangle$

The action of $\Gamma/K = \mathbb{Z}_{4k}$ on the nilmanifold $\mathcal{H}/K$ is given by $\langle h \rangle$:

$$h(x, y, z) = (-x, -y, z + \frac{1}{4k}).$$

The action of $\Gamma/N = \mathbb{Z}_{8k} \times \mathbb{Z}_2$ on the nilmanifold $\mathcal{H}/N$ is given by

$$\langle f, g \rangle : f(x, y, z) = (-x, -y, z + \frac{1}{8k}), \qquad g(x, y, z) = (x, y + \frac{1}{2}, z).$$

The action of $\Gamma/L = \mathbb{Z}_{16k} \times \mathbb{Z}_2 \times \mathbb{Z}_2$ on the nilmanifold $\mathcal{H}/L$ is given by $\langle \phi, \xi, \eta \rangle$:

$$\phi(x, y, z) = (-x, -y, z + \frac{1}{16k}), \xi(x, y, z) = (x + \frac{1}{2}, y, z + \frac{y}{2}), \eta = g.$$

Proof. Let N be a free normal subgroup of Γ such that $G = \Gamma/N$ is abelian. Then $[\Gamma, \Gamma] = \langle t_1^2, t_2^2 \rangle \subset N \subset \langle t_1, t_2, t_3 \rangle$. Suppose N contains both $t_1 t_2^{\ell_1} t_3^{\ell_2}, t_2 t_3^r$. Then N can be represented by an ordered set of generators

$$\langle t_1 t_2^{\ell_1} t_3^{\ell_2}, t_2 t_3^r, t_3^n \rangle$$

by Lemma 2.4. By the right action of $\begin{pmatrix} 1 & 0 \\ -\ell_1 & 1 \end{pmatrix} \in GL(2, \mathbb{Z}) \subset \text{Aut}(\mathcal{H})$ on N, N reduces to $\langle t_1 t_3^\ell, t_2 t_3^r, t_3^n \rangle$. Since N contains t_1^2 and t_2^2, we

have $t_3^{2\ell}$, $t_3^{2r} \in N$. Thus 2ℓ and $2r$ must be multiples of n. Note that $0 \leq \ell, r < n$. Thus $\ell = 0, \frac{n}{2}$, and $r = 0, \frac{n}{2}$. Since $[t_1 t_3^{\ell}, t_2 t_3^{r}] = [t_1, t_2] = t_3^{2k} \in N$, $2k$ must be a multiple of n. Afterwards we shall do only an infranilmanifold M_1 case, i.e., $n = 2k$. The other cases can be done by a similar method as M_1 case which are left to the reader. Therefore the possible normal subgroups are

$$K_1 = \langle t_1, t_2, t_3^{2k} \rangle, \qquad K_2 = \langle t_1, t_2 t_3^{k}, t_3^{2k} \rangle,$$
$$K_3 = \langle t_1 t_3^{k}, t_2, t_3^{2k} \rangle, \qquad K_4 = \langle t_1 t_3^{k}, t_2 t_3^{k}, t_3^{2k} \rangle.$$

Recall that $\left(\begin{bmatrix} 1 & \frac{p}{2} & * \\ 0 & 1 & \frac{q}{2} \\ 0 & 0 & 1 \end{bmatrix}, \mathrm{GL}(2, \mathbb{Z}) \right) \in N_{\mathrm{Aff}(\mathcal{H})}(\Gamma)$, for $p, q \in \mathbb{Z}$ and $pq \in 2\mathbb{Z}$. It is not hard to see $K_2 \underset{R}{\sim} K_1$, $K_3 \underset{R}{\sim} K_1$ and $K_4 \underset{R}{\sim} K_1$ by using $\begin{bmatrix} 1 & -k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & -k & 0 \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix}$ respectively. Thus we get $K = K_1 = \langle t_1, t_2, t_3^{2k} \rangle = \langle t_1, t_2, \alpha^{4k} \rangle$. Therefore there exists only one $\mathbb{Z}_{4k} = \Gamma/K$ free action on the nilmanifold $\mathcal{H}/K$ which yields an infra-nilmanifold homeomorphic to $\mathcal{H}/\Gamma$.

Suppose N does not contain $t_2 t_3^{r}$, but $t_1 t_2^{\ell} t_3^{s}$. Then, since $t_2^2 \in N$, N can be represented by $\langle t_1 t_2^{\ell} t_3^{s}, t_2^2, t_3^n \rangle$. Since

$$[t_1 t_2^{\ell} t_3^{s}, t_2^2] = [t_1, t_2^2] = [t_1, t_2]^2 = t_3^{4k},$$

$4k$ must be a multiple of n. Thus $n = 4k$. Assume $\ell = 1$. Then $N = \langle t_1 t_2 t_3^{s}, t_2^2, t_3^{4k} \rangle$. Since $t_1^2 \in N$, $t_3^{2s} \in N$. Thus we have $s = 0$ or $2k$. Similar result can be obtained for the case ℓ is an even integer. Therefore the possible normal subgroups are

$$N_1 = \langle t_1 t_2, t_2^2, t_3^{4k} \rangle, \quad N_2 = \langle t_1 t_2 t_3^{2k}, t_2^2, t_3^{4k} \rangle,$$
$$N_3 = \langle t_1, t_2^2, t_3^{4k} \rangle, \quad N_4 = \langle t_1 t_3^{2k}, t_2^2, t_3^{4k} \rangle.$$

Note that $N_i(i = 1, 2, 4)$ is affinely congugate to N_3 by an element of $N_{\text{Aff}(\mathcal{H})}(\Gamma)$. The case when $t_2 t_3^r \in N$, $t_1 t_2^\ell t_3^s \notin N$ can be done similarly.

Lastly, suppose both $t_2 t_3^r, t_1 t_2^\ell t_3^s \notin N$. Then N must be $\langle t_1^2,\ t_2^2,\ t_3^n \rangle$. Since $[t_1^2,\ t_2^2] = t_3^{8k} \in N$, we have $n = 8k$.

In this case, $\Gamma/L = G$ is of the form $\mathbb{Z}_{16k} \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

The realization of the action of $G \cong \Gamma/N$ on the nilmanifold $\mathcal{H}/N$, as an affine action on the standard nilmanifold, is easy provided that we follow the "Realization" procedure. For example, let $N = \langle t_1,\ t_2^2,\ \alpha^{8k} \rangle$. Since G is generated by the images of α and t_2, it is enough to calculate conjugations of α and t_2 by (I, B), where $B = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \in \text{Aut}(\mathcal{H})$.

For $\alpha = (a, A) = \left(\begin{bmatrix} 1 & 0 & \frac{1}{4k} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right)$, we have

$$(I, B)(a, A)(I, B)^{-1} = (B(a), A) = \left(\begin{bmatrix} 1 & 0 & \frac{1}{8k} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ A \right).$$

It acts on $\mathcal{H}$ by

$$\left(\begin{bmatrix} 1 & 0 & \frac{1}{8k} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ A \right) \cdot \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -x & z + \frac{1}{8k} \\ 0 & 1 & -y \\ 0 & 0 & 1 \end{bmatrix}.$$

Therefore if $f : \mathcal{H} \to \mathcal{H}$ is the map generated by α, then

$$f(x, y, z) = (-x, -y, z + \frac{1}{8k}).$$

For $t_2 = (e_2, I)$,

$$(I, B)\, (e_2, I)\, (I, B)^{-1} = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix},\ I \right).$$

Thus, for the pair (N, Γ), we get an affine action of $G = \Gamma/N \approx \mathbb{Z}_{8k} \times \mathbb{Z}_2$ on the standard nilmanifold:

$$f(x, y, z) = (-x, -y, z + \frac{1}{8k}), \qquad g(x, y, z) = (x, y + \frac{1}{2}, z).$$

The other cases are left to the reader.

REFERENCES

1. L.S. Charlap, *Bieberbach Groups and Flat Manifolds*, Springer-Verlag, Universitext, 1986.
2. K. Dekimpe, P. Igodt, S. Kim and K.B. Lee, *Affine structures for closed 3-dimensional manifolds with nil-geometry*, Quart. J. Math. Oxford (2) **46** (1995), 141–167.
3. W. Heil, *On P^2-irreducible 3-manifolds*, Bull. Amer. Math. Soc. **75** (1969), 772–775.
4. W. Heil, *Almost sufficiently large Seifert fiber spaces*, Michigan Math. J. **20** (1973), 217-223.
5. J. Hempel, *Free cyclic actions of $S^1 \times S^1 \times S^1$*, Proc. A.M.S. **48**, **1** (1975), 221–227.
6. K.B. Lee, *There are only finitely many infra-nilmanifolds under each manifold*, Quart. J. Math. Oxford (2) **39** (1988), 61–66.
7. K.B. Lee and F. Raymond, *Rigidity of almost crystallographic groups*, Contemporary Math. **44** (1985), 73–78.
8. K.B. Lee and J.K. Shin, *Free actions of finite abelian groups on the 3-dimensional nilmanifold*, (in preparation).
9. K.B. Lee, J.K. Shin and Y. Shoji, *Free actions of finite abelian groups on the 3-Torus*, Top. its Appl. **53** (1993), 153–175.
10. P. Orlik, *Seifert Manifolds*, Springer Lecture Notes in Math. 291, 1972.
11. F. Waldhausen, *On irreducible 3-manifolds which are sufficiently large*, Ann. of Math. **87** no. 2 (1968), 56–88.
12. J. Wolf, *Spaces of Constant Curvatures*, Publish or Perish, 1974.

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