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THE CORONA THEOREM FOR BOUNDED FUNCTIONS IN DIRICHLET SPACE

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ABSTRACT. In this paper we prove that the corona theorem for the algebra $H^\infty(\mathbf{D}) \cap D(\mathbf{D})$. That is, we prove that $\mathcal{M}\backslash\overline{\mathbf{D}}$ is an empty set where $\mathcal{M}$ is the maximal ideal space of the given algebra.

1. Preliminary

Let $\mathbf{D}$ be the unit disk in the complex plane $\mathbf{C}$. As usual $H^\infty(\mathbf{D})$ is the set of all bounded analytic functions on $\mathbf{D}$. And $D(\mathbf{D})$ is the set of all analytic functions such that f' is square integrable with respect to the usual area measure. We know that $H^\infty(\mathbf{D}) \bigcap D(\mathbf{D})$ forms a Banach Algebra with the norm given by $\|f\| = \|f\|_\infty + \|f\|_{D(\mathbf{D})}$ where

$$\|f\|_\infty = \sup\{|f(z)| : z \in \mathbf{D}\}$$

and

$$\|f\|^2_{D(\mathbf{D})} = \int_{\mathbf{D}} |f'(z)|^2 dA(z).$$

The corona theorem for $H^\infty(\mathbf{D})$ was originally proved by L.Carleson [2] in 1962. Later P.Wolff gave a nice simpler proof using Littlewood-Paley integrals in 1979(see, for example, [3, Chapter 8]). This theorem was generalized for $H^\infty(\Omega)$ where Ω is a finitely connected domain and also for various subalgebras of $H^\infty(\mathbf{D})$. The corona problem can be

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stated as follows. Let Ω be a domain in $\mathbf{C}$ and let $\mathcal{A}$ be a Banach algebra consisting of analytic functions on Ω. Further we assume that $\mathcal{A}$ contains all constant functions. Suppose $f_1, f_2, ..., f_n \in \mathcal{A}$ satisfies $Max_{1\le i\le n} |f_i(z)| \ge \delta > 0$ for all $z \in \Omega$. (These functions are called the corona data). Then, can we find functions $g_1, g_2, ..., g_n \in \mathcal{A}$ and a constant C depending only on δ , $\|f_1\|, \|f_2\|, ..., and \|f_n\|$ such that $f_1g_1 + f_2g_2 + ... + f_ng_n = 1$ where $\|g_i\| \le C$ for all $i \in \{1, 2, ..., n\}$? (These g_i's are called the corona solutions). It turns out that the corona theorem is equivalent to the fact that Ω is dense in $\mathcal{M}$ where $\mathcal{M}$ is the maximal ideal space of $\mathcal{A}$, that is $\mathcal{M}$ is the set of all multiplicative linear functionals on $\mathcal{A}$. Note that, by identifying each $\omega \in \Omega$ with the point evaluation map $\lambda_\omega : \mathcal{A} \to \mathbf{C}$ defined by $\lambda_\omega(f) = f(\omega)$, we can consider Ω as a subset of $\mathcal{M}$. In this argument, we are using the Gelfand topology on $\mathcal{M}$. The set $\mathcal{M}\setminus\overline{\Omega}$ is called the corona. And the corona theorem says that $\mathcal{M} \setminus \overline{\Omega}$ is an empty set.

2. Main Theorem

In this section, we are going to prove the corona theorem for the algebra $H^\infty(\mathbf{D}) \cap D(\mathbf{D})$.

THEOREM. *Suppose* f_1, $f_2, ...,$ f_n *are functions in* $H^\infty(\mathbf{D})\cap D(\mathbf{D})$ *such that* $\mathrm{Max}_{1\le i\le n}|f_i(z)| \ge \delta > 0$ *for all* $z \in \mathbf{D}$. *Then there exists a constant* $C = C(n, \delta)$ *and* $g_1, g_2, ..., g_n \in H^\infty(\mathbf{D}) \cap D(\mathbf{D})$ *such that*

$$f_1g_1 + f_2g_2 + ... + f_ng_n = 1 \tag{1}$$

and $\|g_i\| \le C$ *for all* $i = 1, 2, ..., n$.

Proof. By a normal family argument, we may assume that the corona data $f_1, f_2, ..., f_n \in H^\infty(\mathbf{D}) \cap D(\mathbf{D})$ are analytic on a neighbourhood of the closed unit disk $\overline{\mathbf{D}}$. So we can also assume that

$\|f_i\| \leq 1$ for all $i = 1, 2, ..., n$. It is clear that

$$\varphi_i(z) = \frac{\overline{f_i(z)}}{\sum_{j=1}^n |f_j(z)|^2}, \qquad i = 1, 2, ..., n \tag{2}$$

are non-analytic solutions of (1). Let $g_i = \varphi_i + \sum_{j=1}^n (b_{i,j} - b_{j,i}) f_j$ where $b_{i,j}$ satisfies $\bar{\partial} b_{i,j} = \varphi_j \bar{\partial} \varphi_j$. Then it is easy to check that $\bar{\partial} g_i = 0$ and $g_1, g_2, ..., g_n$ satisfies (1) except the norm conditions. Here we are using the following notations:

$$\begin{aligned} \partial h &= \frac{1}{2}\left(\frac{\partial h}{\partial x} - i\frac{\partial h}{\partial y}\right), \\ \bar{\partial} h &= \frac{1}{2}\left(\frac{\partial h}{\partial x} + i\frac{\partial h}{\partial y}\right), \\ |\nabla h|^2 &= 2(|\partial h|^2 + |\bar{\partial} h|^2). \end{aligned} \tag{3}$$

We will also use $\|f\|^2_{D(\mathbf{D})}$ to denote $\int_{\mathbf{D}} |\nabla f|^2 dA$ for a non-analytic function f. Let $1 \leq j,\ k \leq n$. As in the case of $H^\infty(\mathbf{D})$ (see, for example, [3, Chapter 8]), our problem is equivalent to finding a function $b_{j,k}$ on $\mathbf{D}$ such that

$$\bar{\partial} b_{j,k} = \varphi_j \bar{\partial} \varphi_k \tag{4}$$

in $\mathbf{D}$ where φ_j, φ_k are as in (2) and $\|b_{j,k}\| = \|b_{j,k}\|_\infty + \|b_{j,k}\|_{D(\mathbf{D})} \leq C$ for some constant C depending only on δ and n. Let's use b and g instead of $b_{j,k}$ and $\varphi_j \bar{\partial} \varphi_k$. Then (4) becomes

$$\bar{\partial} b = g. \tag{5}$$

We are looking for b satisfying $\|b\| \leq C$ for some constant depending only on δ and n. Direct calculation shows that (as in [3, page 326])

$$|g(z)|^2 \leq \frac{4}{\delta^6} \sum_{i=1}^n |f_i{}'(z)|^2. \tag{6}$$

In order to find a solution of (5) with $\|b\|_\infty < C_1(n,\delta)$, it is enough to show that $|g(z)|dA(z)$ is a Carleson measure (see [3, page 320]). Let $Q = Q_{\theta,h} = \{z \in \mathbf{D} : |z| \geq 1-h \ \ and \ \ \theta - h \leq Arg\, z \leq \theta + h\}$ for $0 < h < 1$. From (6), we have

$$\begin{aligned}\int_Q |g(z)|dA(z) &\leq \left(\int_{\mathbf{D}} |g(z)|^2 dA(z)\right)^{\frac{1}{2}} (\text{Area } Q)^{\frac{1}{2}} \\ &\leq \frac{2\sqrt{2}}{\delta^3}\sqrt{n}\, h = C_2(n,\delta)h.\end{aligned}$$

Hence $|g(z)|dA(z)$ is a Carleson measure and so, by [3, Theorem 1.1],

$$\inf\{\|\psi\|_\infty : \bar{\partial}\psi = g\} \leq A_1 C_2(n,\delta) \tag{7}$$

for some absolute constant A_1. Now let

$$H_0(z) = \frac{1}{\pi}\int_{\mathbf{D}} \frac{g(\xi)}{z-\xi} dA(\xi).$$

Then

$$\bar{\partial} H_0 = g \tag{8}$$

in $\mathbf{D}$ (see, for example, [5, page 364]). Hence, by (6),

$$\int_{\mathbf{D}} |\bar{\partial} H_0|^2 dA \leq \frac{4n}{\delta^6}. \tag{9}$$

Furthermore ∂H_0 is the Beurling transform of g. And so, as in ([1], page 411),

$$\int_{\mathbf{D}} |\partial H_0|^2 dA \leq A_2 \int_{\mathbf{D}} |g(z)|^2 dA(z)$$

for some absolute constant A_2. Hence, by (6),

$$\int_{\mathbf{D}} |\partial H_0|^2 \leq A_2 \frac{4n}{\delta^6}.$$

Combining this inequality with (4),and (9), we have

$$\|H_0\|^2_{D(\mathbf{D})} = \int_{\mathbf{D}} |\nabla H_0|^2 dA \le C_3(n,\delta). \tag{10}$$

For $f \in L^2(\partial\mathbf{D})$, let

$$\|f\|^2_{B(\partial\mathbf{D})} = \int_{-\pi}^{\pi} \frac{1}{h^2} \int_{-\pi}^{\pi} |f(e^{i(s+h)}) - f(e^{is})|^2 ds dh.$$

Then it is well known (see [6, page 152]) that there exists an absolute constant A_3 such that

$$\frac{1}{A_3}\int_{\mathbf{D}} |\nabla u|^2 dA(z) \le \|f\|^2_{B(\partial\mathbf{D})} \le A_3 \int_{\mathbf{D}} |\nabla u|^2 dA(z), \tag{11}$$

where u is the Poisson integral of $f \in L^2(\partial\mathbf{D})$. By (10), we see that $|\nabla H_0(z)| dA(z)$ is a Carleson measure and so, by [7, Theorem 1.1.2], $\|H_0\|_{BMO} \le C_4$. Here BMO is the set of all functions in $L^2(\partial\mathbf{D})$ which have bounded mean oscillations. Hence $H_0 \in L^2(\partial\mathbf{D})$. Now (10) and (11) implies that

$$\|H_0\|_{B(\partial\mathbf{D})} \le C_5(n,\delta). \tag{12}$$

We know that every solutions of (5) is of the form $b(z) = H_0(z) + h(z)$ where $h(z)$ is in the Disk algebra $A(\mathbf{D})$ (see,for example, [3, page 321]). Let H^2 be the Hardy space. Let's use BMOA($\mathbf{D}$) for the set of functions which are obtained from harmonic extensions of all functions in BMOA = BMO $\cap H^2$. Since $A(\mathbf{D}) \subset BMOA(\mathbf{D})$, the above argument implies that

$$\begin{aligned} &\inf\{\|H_0 - F\|_\infty : F \in BMOA(\mathbf{D})\} \\ &\le \inf\{\|\psi\|_\infty : \bar{\partial}\psi = g\} \le A_1 C_2(n,\delta) \end{aligned} \tag{13}$$

by (7). Peller and Hruscev proved that there exists an unique $F_0 \in BMOA(\mathbf{D})$ satisfying

$$\|H_0 - F_0\|_\infty = \inf\{\|H_0 - F_0\|_\infty : F \in BMOA(\mathbf{D})\} \tag{14}$$

and furthermore

$$\|F_0\|_{B(\partial\mathbf{D})} \leq A_4 \|H_0\|_{B(\partial\mathbf{D})} \tag{15}$$

for some constant A_4 (see [4], page 103). Hence, by (11), (12), and (15), we have

$$\|F_0\|_{D(\mathbf{D})} = \left(\int_{\mathbf{D}} |\nabla F_0|^2 dA\right)^{\frac{1}{2}} \leq \sqrt{A_3}\, \|F_0\|_{B(\partial\mathbf{D})} \leq C_6(n,\delta). \tag{16}$$

Combining (10), and (16), we have

$$\|H_0 - F_0\|_{D(\mathbf{D})} \leq C_7(n,\delta). \tag{17}$$

Finally, combining (13), (14),and (17), we have

$$\|H_0 - F_0\| \leq C(n,\delta).$$

Since $\bar{\partial}(H_0 - F_0) = g$ in $\mathbf{D}$, we complete the proof of our theorem. □

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