JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume 10, August 1997

ON THE MODIFIED HYERS–ULAM–RASSIAS STABILITY OF THE EQUATION $f(x^2 - y^2 + rxy) = f(x^2) - f(y^2) + rf(xy)$

GWANG HUI KIM

ABSTRACT. In this paper, we prove a generalization of the stability by S. M. Jung[3] of the functional equation $f(x^2 - y^2 + rxy) = f(x^2) - f(y^2) + rf(xy)$, which can be considered as a variation of the Hosszu's functional equation.

1 . Introduction

The stability problem of functional equations has been originally raised by S. M. Ulam[6] in 1940. He raised a question : In what conditions does there exist a linear mapping near an approximately additive mapping ?

In 1941, this problem was solved by D. H. Hyers[2]. After a long time, it was improved by Th. M. Rassias[4]. In 1994, this problem was further generalized by P. Găvruta[1]. The Cauchy equation $f(x+y) = f(x)+f(y)$ leads to the Hosszu's equation $f(x+y-xy) = f(x)+f(y)-f(xy)$, and which leads to the function equation $f(x^2 - y^2 + rxy) = f(x^2) - f(y^2) + rf(xy)$.

In this paper, we will prove the modified Hyers-Ulam-Rassias stability of the above last equation by using the Găvruta's method, which is a generalization of Rassias' result. The known stability result of the above equation is the following result by S. M. Jung [3]:

Received by the editors on Jun 30,1997.
1991 *Mathematics Subject Classifications*: Primary 39B72, 47H15.
Key words and phrases: Functional equations, Hyers-Ulam-Rassias stability.

Let $p < 1$, $r > 0$, $r \neq 1$, and $\theta > 0$ be given. Suppose $f : E_1 \to E_2$ to be a mapping such that

$$\|f(x^2 - y^2 + rxy) - f(x^2) + f(y^2) - rf(xy)\| \leq \theta(\|x^2\|^p + \|y^2\|^p)$$

for all $x, y \in E_1$. Then there exists a unique mapping $T : E_1 \to E_2$ satisfying

$$T(x^2 - y^2 + rxy) - T(x^2) + T(y^2) - rT(xy) = 0$$

for all $x, y \in E_1$, and

$$\|f(x) - T(x)\| \leq \frac{2\theta}{|r - r^p|}\|x\|^p \quad \text{for all } x \in E_1,$$

where E_1 is a real normed space for which the multiplication "·" between the elements is defined with $(\alpha x) \cdot (\beta y) = (\alpha\beta)(x \cdot y)$ for all $x, y \in E_1$ and $\alpha, \beta \in R$, and $E_1 = \{x^2 : x \in E_1\} \cup \{-x^2 : x \in E_1\}$. The real space R is an example for the space E_1. Also let E_2 be a real Banach space.

2. Result

Let E_1 and E_2 be those in section 1, and define a mapping $\varphi : E_1 \times E_1 \longrightarrow [0, \infty)$ such that

(1) $$\tilde{\varphi}(x, y) := \sum_{k=0}^{\infty} r^{-k}\varphi(r^{\frac{k}{2}}x, r^{\frac{k}{2}}y) < \infty$$

for all $x, y \in E_1$, where $r > 0$ and $r \neq 1$.

THEOREM. *Let $f : E_1 \longrightarrow E_2$ be a mapping such that*

(2) $$\|f(x^2 - y^2 + rxy) - f(x^2) + f(y^2) - rf(xy)\| \leq \varphi(x, y),$$

for all $x, y \in E_1$, where $r > 0, r \neq 1$. Then there exists a unique mapping $T : E_1 \longrightarrow E_2$ such that, for all $x, y \in E_1$,

$$T(x^2 - y^2 + rxy) - T(x^2) + T(y^2) - rT(xy) = 0 \tag{3}$$

and

$$\|f(x) - T(x)\| \leq \begin{cases} \frac{1}{r}\tilde{\varphi}(z,z), & \text{if } x = z^2, \\ \frac{1}{r}\tilde{\varphi}(z,-z), & \text{if } x = -z^2. \end{cases} \tag{4}$$

Proof. For $x = y$ inequality (2) implies

$$\|f(rx^2) - rf(x^2)\| \leq \varphi(x,x).$$

Thus

$$\|r^{-1}f(rx^2) - f(x^2)\| \leq \frac{1}{r}\varphi(x,x), \quad \text{for all } x \in E_1. \tag{5}$$

Replacing x by $\sqrt{r}x$, inequality (5) gives

$$\|r^{-1}f(r^2x^2) - f(rx^2)\| \leq \frac{1}{r}\varphi(\sqrt{r}x, \sqrt{r}x). \tag{6}$$

From (5) and (6) it follows that

$$\begin{aligned} &\|r^{-2}f(r^2x^2) - f(x^2)\| \\ &\qquad \leq \|r^{-2}f(r^2x^2) - r^{-1}f(rx^2)\| + \|r^{-1}f(rx^2) - f(x^2)\| \\ &\qquad \leq r^{-1}\|r^{-1}f(r^2x^2) - f(rx^2)\| + \frac{1}{r}\varphi(x,x) \\ &\qquad \leq \frac{1}{r}\varphi(x,x) + \frac{1}{r^2}\varphi(\sqrt{r}x, \sqrt{r}x). \end{aligned}$$

Hence

$$\|r^{-2}f(r^2x^2) - f(x^2)\| \leq \frac{1}{r}[\varphi(x,x) + \frac{1}{r}\varphi(\sqrt{r}x, \sqrt{r}x)] \tag{7}$$

for all $x \in E_1$.

Applying an induction argument to n we obtain

$$\|r^{-n}f(r^n x^2) - f(x^2)\| \le \frac{1}{r}\sum_{k=0}^{n-1} r^{-k}\varphi(r^{\frac{k}{2}}x, r^{\frac{k}{2}}x) \tag{8}$$

for all $x \in E_1$. Indeed,

$$\begin{aligned}\|r^{-(n+1)}f(r^{n+1}x^2) - f(x^2)\| &\le \|r^{-(n+1)}f(r^{n+1}x^2) - r^{-1}f(rx^2)\| \\ &\quad + \|r^{-1}f(rx^2) - f(x^2)\|\end{aligned}$$

and with (8) and (5) we obtain

$$\begin{aligned}&\|r^{-(n+1)}f(r^{n+1}x^2) - f(x^2)\| \\ &\qquad \le r^{-1}\frac{1}{r}\sum_{k=0}^{n-1} r^{-k}\varphi(r^{\frac{k+1}{2}}x, r^{\frac{k+1}{2}}x) + \frac{1}{r}\varphi(x,x) \\ &\qquad = \frac{1}{r}\sum_{k=0}^{n} r^{-k}\varphi(r^{\frac{k}{2}}x, r^{\frac{k}{2}}x).\end{aligned}$$

On the other hand, by putting $y = -x$ in (2), we have the following inequality

$$\|f(-rx^2) - rf(-x^2)\| \le \varphi(x, -x).$$

Throughout the same process from (5) to (8), we can easily prove

$$\|r^{-n}f(-r^n x^2) - f(-x^2)\| \le \frac{1}{r}\sum_{k=0}^{n-1} r^{-k}\varphi(r^{\frac{k}{2}}x, -r^{\frac{k}{2}}x) \tag{9}$$

for all $n \in N,\ x \in E_1$.

According to (8), (9) and the properties of the space E_1, we conclude

$$\begin{aligned}\|r^{-n}f(r^n x) - f(x)\| &= \begin{cases} \|r^{-n}f(r^n z^2) - f(z^2)\| \\ \|r^{-n}f(-r^n z^2) - f(-z^2)\| \end{cases} \\ &\le \begin{cases} \dfrac{1}{r}\displaystyle\sum_{k=0}^{n-1} r^{-k}\varphi(r^{\frac{k}{2}}z, r^{\frac{k}{2}}z) \\ \dfrac{1}{r}\displaystyle\sum_{k=0}^{n-1} r^{-k}\varphi(r^{\frac{k}{2}}z, -r^{\frac{k}{2}}z) \end{cases}\end{aligned} \tag{10}$$

for all $n \in N$ and each $x = z^2$ or $x = -z^2 \in E_1$.

Given $x \in E_1$, we now define

$$T(x) := \begin{cases} \lim_{n\to\infty} r^{-n} f(r^n x) & \text{for } r > 1 \\ \lim_{n\to\infty} r^n f(r^{-n} x) & \text{for } 0 < r < 1. \end{cases}$$

Let $n > m > 0$. Using (10), we then obtain, for $r > 1$,

$$\begin{aligned} \|r^{-n} f(r^n x) - r^{-m} f(r^m x)\| &= r^{-m} \|r^{-(n-m)} f(r^{n-m} r^m x) - f(r^m x)\| \\ &\le r^{-m} \frac{1}{r} \sum_{k=0}^{n-m-1} r^{-k} \varphi(r^{\frac{k+m}{2}} z, r^{\frac{k+m}{2}} z) \\ &= \frac{1}{r} \sum_{p=m}^{n-1} r^{-p} \varphi(r^{\frac{p}{2}} z, r^{\frac{p}{2}} z), \end{aligned}$$

for each $x = z^2$, with $x, z \in E_1$. Taking the limit as $m \to \infty$ we obtain

$$\|r^{-n} f(r^n x) - r^{-m} f(r^m x)\| \to 0.$$

Similarly, we have for $0 < r < 1$

$$\|r^n f(r^{-n} x) - r^m f(r^{-m} x)\| \to 0$$

as $m \to \infty$.

Hence the above two conditions imply that both sequences

$$\{r^{-n} f(r^n x)\}_{n\in N}, \qquad \text{for } r > 1$$

and

$$\{r^n f(r^{-n} x)\}_{n\in N}, \qquad \text{for } 0 < r < 1$$

are cauchy sequences. Since E_2 is the Banach space, the value of T is defined for all $x \in E_1$. We consider the case that $r > 1$. The other case can be proved similarly . We claim that T satisfies (3).

By using (2)

$$\begin{aligned}\|T(x^2 - y^2 + rxy) - T(x^2) + T(y^2) - rT(xy)\| \\ &= \lim_{n\to\infty} \frac{1}{r^n}\|f(r^n x^2 - r^n y^2 + r^{n+1}xy) \\ &\qquad - f(r^n x^2) + f(r^n y^2) - rf(r^n xy)\| \\ &\le \lim_{n\to\infty} \frac{1}{r^n}\varphi(\sqrt{r^n}x, \sqrt{r^n}y) \\ &= 0.\end{aligned}$$

Hence, we arrive that (3) holds for all $x, y \in E_1$.

To prove (4), taking the limit in (10) as $n \to \infty$

$$\|T(x) - f(x)\| \le \begin{cases} \frac{1}{r}\tilde{\varphi}(z,z), & \text{for } x = z^2, \\ \frac{1}{r}\tilde{\varphi}(z,-z), & \text{for } x = -z^2. \end{cases}$$

It remains to show that T is unique.

Assume that $T' : E_1 \longrightarrow E_2$ were another such mapping with (3) and (4) satisfied. It follows from (3) with some calculation that

$$T(r^n x) = r^n T(x) \quad \text{and} \quad T'(r^n x) = r^n T'(x) \tag{11}$$

for every $n \in N$ and each $x \in E_1$. Further, assume $T'(x) \neq T(x)$ for some $x \in E_1$. Using (4) and (11), we have

$$\begin{aligned}\|T(x) - T'(x)\| &= \|r^{-n}T(r^n x) - r^{-n}T'(r^n x)\| \\ &\le r^{-n}(\|T(r^n x) - f(r^n x)\| + \|f(r^n x) - T'(r^n x)\|) \\ &\le r^{-n}(\frac{2}{r}\tilde{\varphi}(r^n x, r^n x)) \to 0 \quad \text{as } n \to \infty,\end{aligned}$$

which leads to the contradiction. Hence the proof of the theorem is complete. □

3. Application

For the application of Jung's theorem, let G be a normed linear space and define $H : R_+ \times R_+ \to R_+$ and $\varphi_0 : R_+ \to R_+$ such that

$$\begin{aligned}
\varphi_0(\lambda) &> 0, && \text{for all } \lambda > 0,\\
\varphi_0(r) &< r, &&\\
\varphi_0(st) &\leq \varphi_0(s)\varphi_0(t), && \text{for all } s,t \in R_+,\\
H(\lambda s, \lambda t) &\leq \varphi_0(\lambda)H(s,t), && \text{for all } s,t \in R_+, \lambda > 0,\\
H(\|x^2\|, \|y^2\|) &= \theta(\|x\|^p + \|y\|^p), && \text{for all } x,y \in E_1.
\end{aligned}$$

We take in our theorem $\varphi(x,y) = H(\|x^2\|, \|y^2\|)$. Then

$$\begin{aligned}
\varphi(r^{\frac{k}{2}}x, r^{\frac{k}{2}}y) &= H(\|(r^{\frac{k}{2}}x)^2\|, \|(r^{\frac{k}{2}}y)^2\|)\\
&= H(r^k\|x^2\|, r^k\|y^2\|)\\
&\leq \varphi_0(r^k)H(\|x^2\|, \|y^2\|)\\
&\leq (\varphi_0(r))^k H(\|x^2\|, \|y^2\|)
\end{aligned}$$

and because $\varphi_0(r) < r$ we have

$$\begin{aligned}
\tilde{\varphi}(x,y) &\leq \sum_{k=0}^{\infty}(r^{-k})(\varphi_0(r))^k H(\|x^2\|, \|y^2\|)\\
&= \frac{1}{1-(\varphi_0(r)/r)}H(\|x^2\|, \|y^2\|),
\end{aligned}$$

and the relation (4) becomes

$$\begin{aligned}
\|f(x) - T(x)\| &\leq \begin{cases} \dfrac{1}{r}\tilde{\varphi}(z,z), & \text{if } x = z^2,\\[2mm] \dfrac{1}{r}\tilde{\varphi}(z,-z), & \text{if } x = -z^2 \end{cases}\\
&\leq \frac{1}{r-\varphi_0(r)}H(\|z^2\|, \|z^2\|)\\
&= \frac{1}{r-\varphi_0(r)}2\theta\|x\|^p.
\end{aligned}$$

References

1. P. Găvruta, *A Generalization of the Hyers-Ulam-Rassias Stability of Approximately Additive Mappings*, J. Math. Anal. Appl. **184** (1994), 431-436..
2. D. H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U. S. A. **27** (1941), 222-224..
3. S. M. Jung, *On the Hyers-Ulam-Rassias stability of the equation* $f(x^2-y^2+rxy)=f(x^2)-f(y^2)+rf(xy)$, Bull. Korean Math. Soc. **33 No.4** (1996), 513-519.
4. Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297-300.
5. Th. M. Rassias and P. Šemrl, *On the Behavior of Mappings which do not satisfy Hyers-Ulam Stability*, Proc. Amer. Math. Soc. **114 No 4** (1992), 989-993.
6. S. M. Ulam, *"Problems in Modern Mathematics" Chap. VI, Science eds., Wiley, New York,.*

Gwang Hui Kim
Department of Mathematics
University of Kangnam
Suwon 449-702, Korea

E-mail: ghkim@kns.kangnam.ac.kr