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DERIVATIONS MAPPING INTO THE RADICAL

HAK-MAHN KIM

ABSTRACT. In this paper we prove that if D is a continuous derivation on a noncommutative complex Banach algebra A and $[D(x), x]$ is idempotent for every $x \in A$, then D maps A into its radical.

1. Introduction

We write $[x, y]$ for $xy - yx$. An additive mapping D from a Banach algebra A to A is called a derivation if $D(xy) = D(x)y + xD(y)$ holds for all $x, y \in A$. An idempotent of an algebra is an element x of the algebra with $x^2 = x$. The purpose of this paper is to present some conditions which imply that a continuous derivation of a Banach algebra maps the algebra into its radical. The classical result of Singer and Wermer asserts that in a commutative Banach algebra this holds for every continuous derivation. There have been various ways to extend this theorem to noncommutative algebras. Our theorem depends on the classical result of Posner.

LEMMA 1.1. ([3, Lemma 3.]) *Let D be a derivation of a noncommutative prime ring R. If for every $x \in R$, the element $[D(x), x] = 0$, then $D = 0$.*

THEOREM 1.2. ([1, Theorem 3]) *Let D be a continuous derivation of a Banach algebra A. If $[D(x), x]^2$ lies in rad(A) for all $x \in A$, then D maps A into rad(A).*

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2. Main Results

By Theorem 1.2, we obtain that if D is a continuous derivation of a noncommutative semisimple Banach algebra A and $[D(x), x]^2 = 0$ for all $x \in A$, then $D = 0$. Our next theorem is related to this result and Theorem 1.2.

THEOREM 2.1. *Let D be a continuous derivation on a noncommutative complex Banach algebra A. If $[D(x), x]$ is idempotent for every x in A, then D maps A into rad(A).*

Proof. Sinclair [2] has proved that any continuous linear derivation on a Banach algebra leaves the primitive ideals of the algebra invariant. Hence for any primitive ideal $P \subseteq A$ one can introduce a derivation $D_P : A/P \longrightarrow A/P$, where A/P is a factor Banach algebra, by $D_P(\bar{x}) = \overline{D(x)}$, $\bar{x} = x + P$. The assumption of the theorem $[D(x), x]^2 = [D(x), x]$, $x \in A$ gives $[D_P(\bar{x}), \bar{x}]^2 = [D_P(\bar{x}), \bar{x}]$, $\bar{x} \in A/P$. The factor algebra A/P is prime, since P is a primitive ideal. Thus it suffices to show that a continuous derivation D on a primitive Banach algebra A satisfying $[D(x), x]^2 = [D(x), x]$, $x \in A$, is a zero mapping.

Let us introduce a mapping $B : A \times A \longrightarrow A$ by the relation

$$B(x, y) = [D(x), y] + [D(y), x], \ x, y \in A.$$

Obviously, the mapping B is a symmetric bilinear mapping. We shall write $f(x)$ for $B(x, x)$. Then $f(x) = 2[D(x), x]$. Then by assumption $f(x)^2 - 2f(x) = 0$ for all $x \in A$. Now, for $x, y \in A$, $\lambda \in \mathbb{C}$, we have

$$\begin{aligned}
0 &= f(x + \lambda y)^2 - 2f(x + \lambda y) \\
&= (f(x) + 2\lambda B(x, y) + \lambda^2 f(y))^2 - 2(f(x) + 2\lambda B(x, y) + \lambda^2 f(y)) \\
&= \lambda(2f(x)B(x, y) + 2B(x, y)f(x) - 4B(x, y)) \\
&\quad + \lambda^2(f(x)f(y) + f(y)f(x) + 4B(x, y)^2 - 2f(y)) \\
&\quad + \lambda^3(2B(x, y)f(y) + 2f(y)B(x, y)) + \lambda^4 2f(y).
\end{aligned}$$

Substituting $\lambda = 1,\ \lambda = -1$, we get, for each $x, y \in A$,

$$
\begin{aligned}
0 &= (2f(x)B(x,y) + 2B(x,y)f(x) - 4B(x,y)) \\
&\quad + (2B(x,y)f(y) + 2f(y)B(x,y)).
\end{aligned}
\tag{1}
$$

Using $\lambda = i,\ \lambda = -i$, we obtain, for each $x, y \in A$,

$$
\begin{aligned}
0 &= (2f(x)B(x,y) + 2B(x,y)f(x) - 4B(x,y)) \\
&\quad - (2B(x,y)f(y) + 2f(y)B(x,y)).
\end{aligned}
\tag{2}
$$

Combining (1) with (2), we arrive at, for each $x, y \in A$,

$$f(x)B(x,y) + B(x,y)f(x) - 2B(x,y) = 0$$

and

$$B(x,y)f(y) + f(y)B(x,y) = 0.$$

Since B is symmetric, the last two relations imply that $B(x,y) = 0$ for every $x, y \in A$. Hence $[D(x), y] + [D(y), x] = 0$ for all $x, y \in A$. In particular, $[D(x), x] = 0$ for all $x \in A$ and by Lemma 1.1, we have $D = 0$. This proves the theorem. □

COROLLARY 2.2. *Let A be a noncommutative semisimple Banach algebra over $\mathbb{C}$ and D a derivation of A. If $[D(x), x]$ is idempotent for every x in A, then $D = 0$.*

Proof. By the result of B. E. Johnson and A. M. Sinclair [2] any derivation on a semisimple Banach algebra is continuous. Hence $D = 0$. □

Neglecting the additional assumption concerning the characteristic of the ring, we can say that the following theorem generalizes Posner's Theorem [3, Lemma 3].

COROLLARY 2.3. *Let R be a noncommutative prime ring of characteristic not 2, 3 and D a derivation of R such that $[D(x),x]^2 = [D(x),x]$ for all $x \in R$. Then $D = 0$.*

Proof. We shall use the notation B, f as in the proof of Theorem 2.1. Thus for every $x, y \in R$, we have by (1)

$$\begin{aligned} 0 &= f(x)B(x,y) + B(x,y)f(x) - 2B(x,y) \\ &\quad + B(x,y)f(y) + f(y)B(x,y). \end{aligned}$$

Taking $y = x$, we obtain $0 = 2f(x)^2 - f(x)$. Since $f(x)^2 = 2f(x)$, the last relation reduces to $0 = 3f(x)$ and so $0 = [D(x),x]$ for all $x \in R$. By Lemma 1.1 we have $D = 0$. □

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HAK-MAHN KIM
DEPARTMENT OF MATHEMATICS
CHUNGNAM NATIONAL UNIVERSITY
TAEJON 305-764, KOREA