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A RESULT CONCERNING DERIVATIONS IN NONCOMMUTATIVE BANACH ALGERAS

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ABSTRACT. The purpose of this paper is to prove the following result: Let A be a noncommutative semisimple Banach algebra. Suppose that $D : A \to A$, $G : A \to A$ are linear derivations such that

$$[G(x), x]D(x) = D(x)[G(x), x] = 0, \ [D(x), G(x)] = 0$$

hold for all $x \in A$. In this case either $D = 0$ or $G = 0$.

1. Introduction

Throughout this paper R will represent an associative ring with center $Z(R)$. We write $[x, y] = xy - yx$ and use the identities $[xy, z] = [x, z]y + x[y, z]$, $[x, yz] = [x, y]z + y[x, z]$. An additive mapping $D : R \to R$ is called a derivation if $D(xy) = D(x)y + xD(y)$, $x, y \in R$. A derivation D is inner if there exists $a \in R$ such that $D(x) = [a, x]$ holds for $x \in R$. Recall that R is prime if $aRb = (0)$ implies that either $a = 0$ or $b = 0$. B. E. Johnson and A. M. Sinclair [2] have proved that any linear derivation on a semisimple Banach algebra is continuous. A result of I. M. Singer and J. Wermer [5] states that any continuous linear derivation on a commutative Banach algebra maps the algebra into its radical. Combining these two results one obtains that there are no nonzero linear derivations on a commutative semisimple Banach algebra. In a very recent paper M. P. Thomas [6] has generalized the Singer-Wermer Theorem by proving that any

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linear derivation on a commutative Banach algebra maps the algebra into its radical. Obviously, this result also implies that any linear derivation on a commutative semisimple Banach algebra is zero. Since all linear derivations on a commutative semisimple Banach algebras are zero, it seems natural to ask, under what additional assumptions a linear derivation on a noncommutative semisimple Banach algebra is zero. In Theorem 2.1 we give a partial answer to the above question.

2. Main Results

THEOREM 2.1. *Let A be a noncommutative semisimple Banach algebra. Suppose that $D : A \to A$, $G : A \to A$ are linear derivations such that*

$$[G(x), x]D(x) = D(x)[G(x), x] = 0, \ [D(x), G(x)] = 0$$

hold for all $x \in A$. In this case either $D = 0$ or $G = 0$.

For the proof of Theorem 2.1 we shall need the following purely algebraic result which might be of some independent interest.

THEOREM 2.2. *Let R be a noncommutative prime ring of characteristic different from two and three. Suppose that $D : R \to R$, $G : R \to R$ are derivations such that*

$$[G(x), x]D(x) = D(x)[G(x), x] = 0, \ [D(x), G(x)] = 0$$

hold for all $x \in R$. In this case either $D = 0$ or $G = 0$.

Proof. We intoduce a mapping $B : R \times R \to R$ by the relation

$$B(x, y) = [G(x), y] + [G(y), x], \ x, y \in R. \tag{1}$$

Obviously, the mapping $B(x, y)$ is symmetric (i.e., $B(x, y) = B(y, x)$ for all $x, y \in R$) and additive in both arguments. A routine calculation shows that the relation

$$B(xy, z) = B(x, z)y + xB(y, z) + G(x)[y, z] + [x, z]G(y) \tag{2}$$

holds for all $x, y, z \in R$. We shall write $f(x)$ for $B(x, x)$. Then

$$f(x) = 2[G(x), x],\ x \in R. \tag{3}$$

The mapping f satisfies the relation

$$f(x+y) = f(x) + f(y) + 2B(x, y),\ x, y \in R. \tag{4}$$

Throughout the paper we shall use the mapping B and f, as well as the relations (1),(2),(3) and (4) without specific reference. The assumption of Theorem 2.2 can now be written as follows

$$f(x)D(x) = 0,\ x \in R \tag{5-a}$$

and

$$D(x)f(x) = 0,\ x \in R. \tag{5-b}$$

The linearization of (5-a) (i.e., substitution of $x+y$ instead of x) gives

$$\begin{aligned} 0 &= (f(x) + f(y) + 2B(x, y))(D(x) + D(y)) \\ &= f(y)D(x) + 2B(x, y)D(x) + f(x)D(y) \\ &\quad + 2B(x, y)D(y), \end{aligned} \tag{6}$$

for all $x, y \in R$. Replacing x by $-x$ in (6), we obtain

$$f(x)D(y) + 2B(x, y)D(x) = 0,\ x, y \in R. \tag{7}$$

Replace y by $yG(x)$ in (7), then we have

$$[f(x), y]D(G(x)) + 2[y, x]G^2(x)D(x) = 0,\ x, y \in R. \tag{8}$$

Replace y by yz in (8), then we have by (8)

$$[f(x), y]zD(G(x)) + 2[y, x]zG^2(x)D(x) = 0,\ x, y, z \in R. \tag{9}$$

In particular for $y = f(x)$, we obtain

$$[f(x), x]zG^2(x)D(x) = 0, \ x, z \in R. \tag{10}$$

We intend to prove that

$$G^2(x)D(x) = 0, \ x \in R. \tag{11}$$

Suppose on the contrary that $G^2(a)D(a) \neq 0$ for some $a \in R$. Then it follows from (10) that $[f(a), a] = 0$ by primeness of R. Replace x by a and y by az in (7), then by (5-a) we have

$$f(a)zD(a) + G(a)[z, a]D(a) = 0, \ z \in R. \tag{12}$$

Put $z = G(a)x$ in (12). Then

$$\begin{aligned} 0 &= f(a)G(a)xD(a) + G(a)[G(a)x, a]D(a) \\ &= f(a)G(a)xD(a) + G(a)[G(a), a]xD(a) + G(a)^2[x, a]D(a). \end{aligned}$$

Hence

$$2f(a)G(a)xD(a) + G(a)f(a)xD(a) + 2G(a)^2[x, a]D(a) = 0 \tag{13}$$

for all $x \in R$. On the other hand the left multiplication of the relation (12) by $G(a)$ and letting $z = x$ give

$$G(a)f(a)xD(a) + G(a)^2[x, a]D(a) = 0, \ x \in R. \tag{14}$$

From (13) and (14), we have

$$2f(a)G(a) = G(a)f(a), \tag{15}$$

since R is prime and $G^2(a)D(a) \neq 0$ implies $D(a) \neq 0$. In the same fashion starting from (5-b), we have

$$2G(a)f(a) = f(a)G(a). \tag{16}$$

From (15) and (16), we have

$$f(a)G(a) = 0,\ G(a)f(a) = 0.$$

Let $x = a$ and $y = G(a)$ in (9). Then by primeness of R, $f(a) = 0$. Thus for $x = a$ the relation (9) gives

$$[y,a]zG^2(a)D(a) = 0,\ y,z \in R,$$

which implies $a \in Z(R)$. We have therefore proved that $G^2(x)D(x) = 0$ in case $x \notin Z(R)$. It remains to prove that $G^2(x)D(x) = 0$ also in the case when $x \in Z(R)$. Let therefore x be from $Z(R)$ and $y \notin Z(R)$. We have $x + y \notin Z(R)$. We know that

$$G^2(y)D(y) = 0,\ G^2(x+y)D(x+y) = 0.$$

Then

$$G^2(x)D(x) + G^2(x)D(y) + G^2(y)D(x) = 0. \tag{17}$$

Replace x by $-x$ in (17), then

$$G^2(x)D(x) - G^2(x)D(y) - G^2(y)D(x) = 0. \tag{18}$$

From (17) and (18) it follows that $G^2(x)D(x) = 0$, which completes the proof of (11). By (11), $G^2(x+y)D(x+y) = 0$ for all $x, y \in R$, hence

$$G^2(x)D(y) + G^2(y)D(x) = 0,\ x,y \in R. \tag{19}$$

The substitution yz of y in (19) gives

$$G^2(y)[z, D(x)] + [G^2(x), y]D(z) + 2G(y)G(z)D(x) = 0$$

for all $x, y, z \in R$. The substitution $z = G(x)$ in the above relation gives

$$[G^2(x), y]D(G(x)) = 0, \ x, y \in R. \tag{20}$$

Replace y by yz, then

$$[G^2(x), y]zD(G(x)) = 0, \ x, y, z \in R. \tag{21}$$

Hence

$$[G^2(x), x]zD(G(x)) = 0, \ x, z \in R. \tag{22}$$

Replacing x by $x + y$ in (22), we have

$$\begin{aligned} &[G^2(x), x]zD(G(y)) + [G^2(x), y]zD(G(x)) \\ &+ [G^2(x), y]zD(G(y)) + [G^2(y), x]zD(G(x)) \\ &+ [G^2(y), x]zD(G(y)) + [G^2(y), y]zD(G(x)) = 0 \end{aligned} \tag{23}$$

for all $x, y, z \in R$. The substitution $-x$ for x in (23) gives

$$\begin{aligned} &[G^2(x), x]zD(G(y)) + [G^2(x), y]zD(G(x)) \\ &- [G^2(x), y]zD(G(y)) + [G^2(y), x]zD(G(x)) \\ &- [G^2(y), x]zD(G(y)) - [G^2(y), y]zD(G(x)) = 0 \end{aligned} \tag{24}$$

for all $x, y, z \in R$. From (21), (23) and (24), we have

$$[G^2(x), x]zD(G(y)) + [G^2(y), x]zD(G(x)) = 0, \ x, y, z \in R. \tag{25}$$

Put $z = zD(G(x))t$ in (25). Then

$$[G^2(x), x]zD(G(x))tD(G(y)) + [G^2(y), x]zD(G(x))tD(G(x)) = 0$$

for all $x, y, z, t \in R$. Hence by (22) we have

$$[G^2(y), x]zD(G(x))tD(G(x)) = 0, \; x, y, z, t \in R.$$

By primeness of R, either $[G^2(y), x]zD(G(x)) = 0$ or $D(G(x)) = 0$. In both cases

$$[G^2(y), x]zD(G(x)) = 0.$$

Hence by (25) we have

$$[G^2(x), x]zD(G(y)) = 0, \; x, y, z \in R.$$

Since R is prime, either $[G^2(x), x] = 0$ or $D(G(y)) = 0$. Assume that $[G^2(x), x] = 0$ holds for all $x \in R$. Then we have $G = 0$ as in the proof of [1, Theorem 1]. If $D(G(y)) = 0$ for all $y \in R$, then either $D = 0$ or $G = 0$ by Posner's Theorem [3, Theorem 1]. The proof is complete. □

Theorem 2.1 is in the spirit of result of Vukman [7].

Proof of Theorem 2.1. By the result of B. E. Johnson and A. M. Sinclair [2] any linear derivation on a semisimple Banach algebra is continuous. Sinclair [3] has proved that any continuous linear derivation on a Banach algebra leaves the primitive ideals of the algebra invariant. Hence for any primitive ideal $P \subseteq A$ one can introduce linear derivations $D_p : A/P \to A/P$, $G_p : A/P \to A/P$, where A/P is a factor Banach algebra, by $D_p(\hat{x}) = D(x) + P$, $G_p(\hat{x}) = G(x) + P$, $\hat{x} = x + P$. The assumption of Theorem 2.1

$$[G(x), x]D(x) = D(x)[G(x), x] = 0, \; [D(x), G(x)] = 0, \; x \in A$$

give

$$[G_p(\hat{x}), \hat{x}]D_p(\hat{x}) = D_p(\hat{x})[G_p(\hat{x}), \hat{x}] = 0, \; [D_p(\hat{x}), G_p(\hat{x})] = 0,$$

$\hat{x} \in A/P$. The factor algebra A/P is prime, since P is a primitive ideal. Hence, in case A/P is noncommutative, we have either $D_p = 0$ or $G_p = 0$, since all the assumptions of Theorem 2.2 are fulfilled. In case A/P is a commutative Banach algebra, one can conclude that $D_p = 0$ and $G_p = 0$ since A/P is semisimple and we know that there are no nonzero linear derivations on commutative semisimple Banach algebras. Since A is semisimple, it follows that $D = 0$ or $G = 0$. The proof of Theorem 2.1 is complete. □

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