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UNIQUENESS OF POSITIVE SOLUTIONS FOR PREDATOR-PREY INTERACTING SYSTEMS WITH NONLINEAR DIFFUSION RATES

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ABSTRACT. In general, the positive solution to biological reaction-diffusion equations is not unique. In this paper, we state the sufficient and necessary conditions of the existence of positive solutions, and give and the proof for the uniqueness of positive solutions for a certain elliptic interacting system.

1. Introduction

For the last decade, intensive studies have been pursuing reaction-diffusion equations modeling of various systems in mathematical biology, especially the elliptic steady-states of competitive and predator-prey interacting processes with various boundary conditions.

In this paper, we are interested in the uniqueness and existence of positive solutions to the predator-prey interacting system:

$$(1)\qquad \begin{cases} -\varphi(u)\Delta u = u(f(u) - av^2) & \text{in } \Omega \\ -\psi(v)\Delta v = v(au^2 + g(v)) \\ \dfrac{\partial u}{\partial n} + \kappa u = 0 & \text{on } \partial\Omega \\ \dfrac{\partial v}{\partial n} + \sigma v = 0. \end{cases}$$

Here Ω is a bounded, smooth domain in $\boldsymbol{R}^n$, u, v are the densities of prey and predator, respectively, the functions f, $g \in C^1(\Omega)$ are nonincreasing, and κ, σ, a are positive constants. Moreover, the functions

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φ, ψ are strictly positive and increasing.

In [1], considering a general form of the above system (1), it has been shown that the existence of positive solutions depends on the sign of the first eigenvalue of the Schrödinger-type operator. However, in general, such positive solution is not unique. In section 4, we state the sufficient and necessary conditions of positive solutions to (1) and show the uniqueness of such positive solution to the system (1) with predator-prey interactions. It is known that for the other two interacting models, i.e., for competing and symbiotic interacting systems, the positive solution is not unique. For example, the number of positive solutions for symbiotic interactions depends on the number of the equilibria of elliptic systems.(See [5])

2. Fixed Point Index

In this section, first we define a fixed point index for compact maps which is defined in the positive cone.

Let (E, P) be an arbitrary odered Banach space with its usual positive cone P. For every open subset U of P and every compact map $f : \bar{U} \to P$, define $i(f, U, P) = deg(I - f, U, 0)$ where I is the identity map and $deg(I - f, U, 0)$ denotes the Leray-Schauder degree with respect to zero of the compact vector field $I - f$, which is defined on the closure of the open set $U \subset E$. Then the fixed point index $i(f, U, P)$ is well-defined, if f has no fixed points on ∂U. Denote the fixed point index with respect to P by $i(f, U)$.

Now we state nontrivial fixed point theorem for arbitrary compact maps by making use of the fixed point index. The following two lemmas can be found in Amann[2].

LEMMA 2.1. *Let $f : \bar{P}_\rho \to P$ be a compact map, where $P_\rho = B_\rho(0) \cap P$, $\rho > 0$. If $f(x) \neq \lambda x$ for any $x \in S^+{}_\rho := (\partial B_\rho(0)) \cap P$ and every $\lambda \geq 1$, then $i(f, P_\rho) = 1$.*

LEMMA 2.2. *Let $f : \bar{P}_\rho \to P$ be a compact map such that $f(0) = 0$. Suppose that f has a right derivative ${f'}_+(0)$ at zero such that 1 is not an eigenvalue of ${f'}_+(0)$ to a positive eigenvector. Then there exists a constant $\sigma_0 \in (0, \rho]$ such that for every $\sigma \in (0, \sigma_0]$,*

(i) $i(f, P_\sigma) = 1$ if ${f'}_+(0)$ has no positive eigenvector to an eigenvalue greater than one.

(ii) $i(f, P_\sigma) = 0$ if ${f'}_+(0)$ possesses a positive eigenvector to an eigenvalue greater than one.

3. Linearization

Suppose that φ is strictly positive, nondecreasing and concave down, and f is monotone nonincreasing C^1-function with $f(0) > 0$.

Consider

$$\text{(2)} \qquad \begin{cases} -\varphi(u)\Delta u = uf(u), \\ \dfrac{\partial u}{\partial n} + \kappa u = 0 \quad \text{on } \partial\Omega. \end{cases}$$

The following lemma can be found in [1]. $\lambda_1(A)$ denotes the first eigenvalue of an operator A.

LEMMA 3.1. *If $\lambda_1(\varphi(0)\Delta + f(0)) > 0$, then the equation (2) has a unique positive solution in $C^2(\bar{\Omega})$.*

Let u_0 be the unique positive solution to the equation (2). We shall linearize the equation (2) at $u = u_0 \geq 0$. Define the solution operator S in bounded subset of $C(\bar{\Omega})$ by $S(u) = \bar{u}$, where $\bar{u}$ is the unique positive solution of

$$\text{(3)} \qquad \begin{cases} -\varphi(\bar{u})\Delta\bar{u} + M\bar{u} = uf(u) + Mu, \\ \dfrac{\partial \bar{u}}{\partial n} + \kappa\bar{u} = 0 \quad \text{on } \partial\Omega \end{cases}$$

and M is large positive constant number which makes the right hand side of (3) positive. Observe that $S(u_0) = u_0$. Also define the solution operator S_L of linearization by $S_L w = v$, where v is the unique solution of

$$(4) \qquad \begin{cases} -\varphi(u_0)\Delta v + Mv = wf(s) + w \cdot sf'(s) + Mw, \\ \dfrac{\partial v}{\partial n} + \kappa v = 0 \quad \text{on } \partial\Omega. \end{cases}$$

LEMMA 3.2. *S is Fréchet differentiable at $u = u_0 \in C(\bar{\Omega})$ and $S'(u_0) = S_L$.*

Proof. To show

$$\| S(u_0 + w) - S(w) - S_L(w) \| = o(\| w \|),$$

where the norm is taken in $C(\Omega)$. we replace w by u for convenience. Let $\| u \|_\infty$ be small. Also let $\tilde{u} = S(u_0 + u)$ and $v = S_L(u)$. Then we have:

$$(5) \quad -\Delta\tilde{u} + M\tilde{u}/\varphi(\tilde{u}) = (u_0 + u)f(u_0 + u)/\varphi(\tilde{u}) + M(u_0 + u)/\varphi(\tilde{u}),$$

$$(6) \qquad -\Delta u_0 + Mu_0/\varphi(u_0) = u_0 f(u_0)/\varphi(u_0) + Mu_0/\varphi(u_0),$$

(7)

$$-\Delta v + Mv/\varphi(u_0) = uf(u_0)/\varphi(u_0) + Mu/\varphi(u_0) + u \cdot u_0 f'(u_0)/\varphi(u_0).$$

Denote A and B as follows:

$$\begin{aligned} A = {} & u[f(u_0 + u)/\varphi(\tilde{u}) - f(u_0)/\varphi(u_0)] \\ & + u_0[f(u_0 + u)/\varphi(\tilde{u}) - f(u_0)/\varphi(u_0) \\ & - uf'(u_0)/\varphi(u_0)] + Mu[1/\varphi(\tilde{u}) - 1/\varphi(u_0)] \\ & + Mu_0[1/\varphi(\tilde{u}) - 1/\varphi(u_0)]. \\ B = {} & M\tilde{u}[1/\varphi(u_0) - 1/\varphi(\tilde{u})]. \end{aligned}$$

Subtracting (5) and (6) from (7) provides that

$$\begin{cases} -\varphi(u_0)\Delta(\tilde{u} - u_0 - v) + M(\tilde{u} - u_0 - v) = \varphi(u_0)[A - B], \\ \dfrac{\partial(\tilde{u} - u_0 - v)}{\partial n} + \kappa(\tilde{u} - u_0 - v) = 0 \quad \text{on } \partial\Omega. \end{cases}$$

Noting that as $u \to 0$, $\tilde{u} \to u_0 = S(u_0)$, it is easy to see that

$$\| A - B \| = o(\| u \|) \text{ as } u \to 0$$

and so

$$\| S(u_0 + u) - S(u_0) - S_L(u) \| = \| \tilde{u} - u_0 - v \| = o(\| u \|).$$

□

4. Uniqueness of Positive Solutions

In this section, we consider system (1):

$$\begin{cases} -\varphi(u)\Delta u = u(f(u) - av^2) \\ -\psi(v)\Delta v = v(au^2 + g(v)) \\ \dfrac{\partial u}{\partial n} + \kappa u = 0 \quad \text{on } \partial\Omega \\ \dfrac{\partial v}{\partial n} + \sigma v = 0. \end{cases}$$

Since the system (1) has predator-prey interactions, we expose the assumptions:

(H1) f, g are C^1-functions and nonincreasing with $f(0) > 0$, $g(0) > 0$, and $a > 0$ is a constant.

(H2) There exist positive constants c_1, c_2 such that

$$f(u) < 0 \text{ for } u > c_1,$$

$$g(v) < 0 \text{ for } v > c_2.$$

(H3) φ, ψ are C^1-functions and nondecreasing, concave down in u, v, respectively. Moreover, φ, $\psi \geq \delta > 0$ for some δ.

We denote by $\lambda_{1,\kappa}(A)$, $\lambda_{1,\sigma}(A)$ the first eigenvalue of an operator A under the boundary conditions $\partial\cdot/\partial n + \kappa\cdot = 0$ and $\partial\cdot/\partial n + \sigma\cdot = 0$, respectively. Also denote by u_0 a unique positive solution to

$$\begin{cases} -\varphi(u)\Delta u = uf(u), \\ \dfrac{\partial u}{\partial n} + \kappa u = 0 \quad \text{on } \partial\Omega \end{cases}$$

and by v_0 a unique solution to

$$\begin{cases} -\psi(v)\Delta v = vg(v), \\ \dfrac{\partial v}{\partial n} + \sigma v = 0 \quad \text{on } \partial\Omega. \end{cases}$$

Next we state the existence theorem for system (1) without proof because the theorem is an immediate consequence of Theorem 2 in [1]. So for the proof of this theorem, refer to [1].

THEOREM 4.1. *Suppose the conditions (H1)-(H3) holds.*

(i) If $f(0) \le \lambda_{1,\kappa}(-\varphi(0)\Delta)$, then (1) has no positive solution.

(ii) Suppose $g(0) > \lambda_{1,\sigma}(-\psi(0)\Delta)$. The system (1) has a positive solution if and only if $f(0) - av_0^2 > \lambda_{1,\kappa}(-\varphi(0)\Delta)$.

(iii) Suppose $g(0) \le \lambda_{1,\sigma}(-\psi(0)\Delta)$. The system (1) has a positive solution if and only if $f(0) > \lambda_{1,\kappa}(-\varphi(0)\Delta)$ and $au_0^2 + g(0) > \lambda_{1,\sigma}(-\psi(0)\Delta)$.

We give the uniqueness of positive solutions to our system which is the main result of this paper.

THEOREM 4.2. *The positive solution of our system (1), if it exists, is unique.*

Proof. Let (u, v) be a positive solution to the system (1). Then $u < c_1$ and $v < c_2$, by the strong maximum principle, and $\| u \|_{C^1(\bar\Omega)}$, $\| v \|_{C^1(\bar\Omega)} < \rho_1$ for $\rho_1 > c_1, c_2$. Let $\rho = \rho_1 + 1$ and $E = [C^1(\bar\Omega)]^2$. Define $\mathcal{A} : \bar{P}_\rho := cl\{u \in P : ||u|| < \rho\} \to P$ via

$$\mathcal{A}(u,v) = ((-\varphi(\cdot)\Delta + M)^{-1}[u(f(u) - av^2) + Mu],$$
$$(-\psi(\cdot)\Delta + M)^{-1}[v(au^2 + g(v)) + Mv]),$$

where M is chosen so large that $-c_1 \max\{|f'(u)| : x \in \bar\Omega,\ 0 \le u \le c_1\} - \{|f(c_1) - a{c^2}_2|\} + M > 0$ and $-c_2 \max\{|g'(v)| : 0 \le v \le c_2\} -$

$|g(c_2)| + M > 0$, and M is not an eigenvalue of the eigenvalue problem

$$\begin{cases} -\varphi(\cdot)\Delta\phi = \lambda\phi & \text{in } \Omega, \\ \dfrac{\partial\phi}{\partial n} + \kappa\phi = 0 & \text{on } \partial\Omega. \end{cases}$$

Each coordinate of $\mathcal{A}$ is a compact operator, so $\mathcal{A}$ itself is a compact operator. Note that $(\bar{u}, \bar{v})$ is a nonnegative solution of (1) if and only if it is a fixed point of $\mathcal{A}$. Let $(\bar{u}, \bar{v})$ be a positive fixed point of $\mathcal{A}$. Assume $\lambda \geq 1$. If (ξ_1, ξ_2) is an eigenpair of $\mathcal{A}'(\bar{u}, \bar{v})$(see Lemma 3.2), then we have

$$\mathcal{A}'(\bar{u}, \bar{v}) \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \lambda \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}. \tag{8}$$

Let u_1 be a solution of

$$\begin{cases} -\varphi(u)\Delta u + Mu = \bar{u}f(\bar{u}) + M\bar{u}, \\ \dfrac{\partial u}{\partial n} + \kappa u = 0 \quad \text{on } \partial\Omega \end{cases}$$

and v_1 a solution of

$$\begin{cases} -\psi(v)\Delta v + Mv = \bar{v}g(\bar{v}) + M\bar{v}, \\ \dfrac{\partial v}{\partial n} + \kappa v = 0 \quad \text{on } \partial\Omega. \end{cases}$$

For the simplicity we use the following notations:

$$U_i = (-\varphi(u_i)\Delta + M)^{-1},\ i = 0, 1,$$

$$V_i = (-\psi(v_i)\Delta + M)^{-1},\ i = 0, 1,$$

$$F(\bar{u}, \bar{v}) = f(\bar{u}) - a\bar{v}^2 + \bar{u}f'(\bar{u}) + M,$$

$$G(\bar{u}, \bar{v}) = a\bar{u}^2 + g(\bar{v}) + \bar{v}g'(\bar{v}) + M.$$

Then by lemma 3.2,

$$\mathcal{A}'(\bar{u}, \bar{v}) = \begin{bmatrix} U_0F(\bar{u}, \bar{v}) & U_0[-2a\bar{u}\bar{v}] \\ V_0[2a\bar{u}\bar{v}] & V_0G(\bar{u}, \bar{v}) \end{bmatrix}. \tag{9}$$

From (9) we have that

$$U_0 F(\bar{u},\bar{v})\xi_1 + U_1(-2a\bar{u}\bar{v})\xi_2 = \lambda\xi_1$$

and

$$V_0 2a\bar{u}\bar{v}\xi_1 + V_1 G(\bar{u},\bar{v})\xi_2 = \lambda\xi_2.$$

These imply that

$$-U_1^{-1}\xi_1 + 1/\lambda \cdot F(\bar{u},\bar{v})\xi_1 + 1/\lambda \cdot (-2a\bar{u}\bar{v})\xi_2 = 0 \tag{10}$$

and

$$-V_1^{-1}\xi_2 + 1/\lambda \cdot G(\bar{u},\bar{v})\xi_2 + 1/\lambda \cdot (2a\bar{u}\bar{v})\xi_1 = 0. \tag{11}$$

Thus $\int_\Omega (10)\cdot\xi_1 + \int_\Omega (11)\cdot\xi_2$ provides

$$\begin{aligned}&\int_\Omega [-U_1^{-1}]\xi_1\cdot\xi_1 + 1/\lambda \cdot F(\bar{u},\bar{v})\xi_1\cdot\xi_1 \\ &+ \int_\Omega [-V_1^{-1}]\xi_2\cdot\xi_2 + 1/\lambda \cdot G(\bar{u},\bar{v})\xi_2\cdot\xi_2 = 0.\end{aligned} \tag{12}$$

On the other hand, we get

$$\begin{aligned}&\lambda_{1,\kappa}[-U_1^{-1} + 1/\lambda \cdot F(\bar{u},\bar{v})] \\ &\le \lambda_{1,\kappa}[\varphi(u_1)\Delta + (f(\bar{u}) - a\bar{v}^2 + \bar{u}f'(\bar{u}))] \\ &< \lambda_{1,\kappa}[\varphi(u_1)\Delta + f(\bar{u})] \\ &= 0.\end{aligned}$$

Similarly,

$$\lambda_{1,\sigma}[-V_1^{-1} + 1/\lambda \cdot G(\bar{u},\bar{v})] < 0.$$

Let $T = m(x)\Delta + q(x)$, $m \in C^1(\Omega)$, $q \in L^\infty(\Omega)$. Since the first eigenvalue of T under homogeneous Dirichlet boundary condition is less than or equal to the one under homogeneous Robin condition (see Ch 11 [6]), the left hand side of (12) is less than zero by the variational property of the first eigenvalues. This contradiction shows that $\mathcal{A}'(\bar{u},\bar{v})$ has no eigenvalue greater than or equal to 1, hence $i(\mathcal{A},(\bar{u},\bar{v})) = 1$ by

Lemma 2.2. However, using homotopy invariance and normalization property of the index, one can show that $i(\mathcal{A}, P_\rho) = 1$ using Lemma 2.1. Therefore $(\bar{u}, \bar{v})$ is the only positive solution to our system. □

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REFERENCES

1. I. Ahn, Existence of positive solutions for predator-prey equations with nonlinear diffusion rates, J. of KMS, 31(4) (1994) 545-558.
2. H. Amann, Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces, SIAM Rev. 18(1976), 620-709.
3. A. Leung, Systems of nonlinear partial differential equations: Applications to biology and engineering, Klumer Acad. Pub. Norwell, MA, 1989.
4. L. Li, Coexistence theorems of steady-state for predator-prey interacting systems, Trans. Amer. Math. Soc. 305(1988), 143-166.
5. L. Li and A. Ghoreishi, On positive solutions of general nonlinear elliptic symbiotic interacting systems, Appl. Anal., vol.14(4)(1991), 281-295.
6. J. Smoller, Shock waves and reaction-diffusion equations, Springer-Verlag, New York, 1983.

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