JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume 10, August 1997

ON δ–FRAMES

SEUNG ON LEE, SEOK JONG LEE AND EUN AI CHOI

ABSTRACT. In this paper, we introduce a new class of δ-frames and study its properties. To do so, we introduce δ-filters, almost Lindelöf frames and Lindelöf frames. First, we show that a complete chain or a complete Boolean algebra is a δ-frame. Next, we show that a δ-frame L is almost Lindelöf iff for any δ-filter F in L, $\vee\{x^* : x \in F\} \neq e$. Last, we show that every regular Lindelöf δ-frame is normal and a Lindelöf δ-frame is preserved under a δ-isomorphism which is dense and codense.

1. Introduction

It is well known that the open set lattice $\Omega(X) = \{G : G$ is an open subset of $X\}$, where X is a topological space, is a frame([7]). In $\Omega(X)$, there is no point of X but open sets, so we call the frame $\Omega(X)$ a pointfree topology. The purpose to write this paper is to introduce another class of frames, namely δ-frames, which is a special type of a frame, and study its basic properties. In section 2, 3, we deal with δ-frames, as Lindelöf spaces to compact spaces, instead of frames. We introduce a concept of δ-frames in which the distributivity holds for arbitrary joins and countable meets. In general, $\Omega(X)$ is a frame but not a δ-frame. But a complete chain or a Boolean algebra is a δ-frame. In a frame L, $cov(L)$ is a filter but need not be a δ-filter. But in a δ-frame L, $cov(L)$ is a δ-filter. We define δ-homomorphism and δ-isomorphism, and using these concepts, we show that a Lindelöf δ-frame is preserved under a δ-isomorphism which is dense and codense.

Received by the editors on June 27, 1997.
1991 *Mathematics Subject Classifications*: Primary 06D 54A99 54D99.
Key words and phrases: δ frame, δ filter, Boolian algebra.

In this paper, a partially ordered set is also called a poset. The usual order relation $\leq$ on the set R of real numbers is a partial order. It is the model of partial orders and thus it is customary to denote any partial order on any set by $\leq$. In this paper, $\geq$ is defined by $a \geq b$ iff $b \leq a$; $<$ is defined by $a < b$ iff $a \leq b$ and $a \neq b$. If $\leq$ is a partial order on L, the smallest (largest, resp.) element of L, if it exists, is the element 0 (e, resp.) such that $0 \leq x$ ($x \leq e$, resp.) for each $x \in L$. Smallest (largest, resp.) elements are unique when they exist, by antisymmetry. We call 0 (e, resp.) as the bottom (top, resp.) element of L. Given a poset $(L, \leq)$ and a subset A of L, A is called bounded above (bounded below, resp.) if the set $\{x \in L : a \leq x \text{ for each } a \in A\}$ of upper bounds of A (the set $\{x \in L : x \leq a \text{ for each } a \in A\}$ of lower bounds of A, resp.) is non-empty; the least upper bound of A (written $lub(A)$ or $\vee A$) is the smallest element of the set of upper bounds of A. It may or may not belong to A. When it exists, it is unique. The greatest lower bound of A (written $glb(A)$ or $\wedge A$) is similarly defined. If $A = \{x, y\}$, then $lub(A)$ is denoted by $x \vee y$ and $glb(A)$ is denoted by $x \wedge y$. If $(L, \leq)$ has both the top element e and the bottom element 0, then $lub(\emptyset) = 0$ and $glb(\emptyset) = e$. From now on, we denote a poset $(L, \leq)$ simply as L.

DEFINITION 1.1.

(1) A poset L is called a *lattice* if every finite subset of L has a least upper bound and a greatest lower bound.
(2) A lattice L is called *complete* if every subset A of L has the least upper bound and the greatest lower bound.

PROPOSITION 1.2. *Let L be a lattice. Then the followings are equivalent:*

(1) *Every subset of L has the least upper bound.*
(2) *Every subset of L has the greatest lower bound.*

Proof. See reference [1],[2],[7]. □

DEFINITION 1.3.

(1) A lattice L is called *distributive* if for any $x, y, z \in L$,

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z),$$

or equivalently,

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z).$$

(2) Let L be a lattice and $x \in L$. If $x \vee y = e$ and $x \wedge y = 0$, then y is called a *complement* of x.

(3) If L is a distributive lattice, then every element x of L has at most one complement. If x has the complement, then the complement of x is denoted by x'.

(4) A distributive lattice is called a *Boolean algebra* if every element has a complement.

DEFINITION 1.4. A complete lattice L is called a *frame* (or *complete Heyting algebra*) if for any $a \in L$ and $S \subseteq L$, $a \wedge (\vee S) = \vee\{\, a \wedge s : s \in S \,\}$.

EXAMPLE 1.5. Let X be a set and $\Omega(X)$ is a topology on X, then $(\Omega(X), \leq)$ is a frame, where $A \leq B$ iff $A \subseteq B$ for $A, B \in \Omega(X)$.

DEFINITION 1.6. Let L be a frame.

(1) For A, $B \subseteq L$, A *refines* B if for any $a \in A$, there is $b \in B$ with $a \leq b$, and denoted by $A \leq B$.

(2) For $a, b \in L$, a is *well inside* b if there is $c \in L$ with $a \wedge c = 0$ and $c \vee b = e$, and denoted by $a \prec b$. Equivalently, $a \prec b \iff a^* \vee b = e$, where $a^* = \vee\{x \in L : a \wedge x = 0\}$ is the pseudo complement of a.

(3) For $A \subseteq L$, A is called a *cover* of L if $\vee A = e$. For a frame L, the set of all covers of L is denoted by $\mathrm{cov}(L)$.
(4) L is called *regular* if for any $x \in L, x = \vee\{a \in L : a \prec x\}$.
(5) L is called *normal* if for $a \vee b = e$, there are $u, v \in L$ with $a \vee u = e = b \vee v$ and $u \wedge v = 0$.

REMARK 1.7. Let L be a frame and $a, b \in L$. Then we have:

(1) $a \wedge a^* = 0$, but $a \vee a^* \neq e$ in general. In $\Omega(X)$ $A^* = int(A^c)$ for any $A \in \Omega(X)$, where $int(A^c)$ is the interior of $X - A$. So $A \vee A^* = A \cup int(A^c) \neq X$ unless A is a clopen subset of X.
(2) If $a \vee a^* = e$, then L is a Boolean Lattice. That is, $a^* = a'$([3]).
(3) $a \leq a^{**}$, because $a \wedge a^* = 0$.
(4) $(a \vee b)^* = a^* \wedge b^*$([3]).
(5) $(a \wedge b)^* \neq a^* \vee b^*$ in general. Consider the open set lattice

$$\Omega(X) = \{X, \{t, u\}, \{t\}, \{u\}, \emptyset\},$$

where $X = \{t, u, v\}$. In this case,

$$(a \wedge b)^* = 0^* = e,$$

but

$$a^* \vee b^* = b \vee a = c,$$

where $e = X,\ a = \{t\},\ b = \{u\},\ c = \{t, u\}$, and $0 = \emptyset$.
(6) $(cov(L), \leq)$ is a quasi-ordered set. And for $A, B \in cov(L)$, the greatest lower bound of A and B is of the form $\{a \wedge b : a \in A, b \in B\}$. That is,

$$A \wedge B = \{a \wedge b : a \in A, b \in B\}.$$

DEFINITION 1.8. Let L and M be frames. A map $f : L \longrightarrow M$ is called a *frame homomorphism* if f preserves finite meets and arbitrary joins. That is,

(1) $f(\vee S) = \vee f(S)$ for any $S \subseteq L$.
(2) $f(\wedge F) = \wedge f(F)$ for any $F \in Fin(L)$, where $Fin(L) = \{A \subseteq L : A$ is a finite subset of $L\}$.

DEFINITION 1.9.([6]) Let L be a complete lattice and $F \subseteq L$. Then we say that F is a *filter* (δ-*filter*, resp.) on L if F satisfies the followings:

(1) F does not contain 0.
(2) $F = \uparrow F = \{x \in L : a \leq x$ for some $a \in F\}$. That is, $a \in F$ and $b \geq a$ implies $b \in F$.
(3) For any finite (countable, resp.) subset K of F, $\wedge K \in F$.

REMARK 1.10.

(1) Let F be a filter on a complete lattice L, then $e \in F$ since $\wedge\emptyset = e$; and hence $F \neq \emptyset$.
(2) Every δ-filter is a filter, but a filter need not be a δ-filter. In fact, on the open set lattice $U(R)$ with the usual topology on R, the neighborhood system $N_A = \{M : M$ is an open neighborhood of $A\}$ is not a δ-filter but a filter on $U(R)$, where $A = \{0\}$. Because $int(\cap\{(-1/n, 1/n) : n \in N\}) = \emptyset$ is not in N_A.
(3) In a complete chain $\{1/n : n \in N\} \cup \{0\}$, $F = \{1/n : n \in N\}$ is a filter but not a δ-filter, because $\wedge\{1/n : n \in N\} = 0$ is not in F.
(4) $F = \{x \in L : a \leq x$ and $a \neq 0\}$ is a δ-filter on a complete lattice L, and denoted by $F = [a]$ for some $a \in L$.

DEFINITION 1.11.([5]) A filter F in a frame L is said to be *clustered* if for any cover S of L, $secF \cap S \neq \emptyset$, where $secF = \{a \in L :$ for any $a \in F, a \wedge x \neq 0\}$.

PROPOSITION 1.12. *A filter F in a frame L is clustered if and only if* $\vee\{x^* : x \in F\} \neq e$.

Proof. See reference [5]. □

2. δ-Frames

DEFINITION 2.1. A frame L is called a *δ-frame* if for any $a \in L$ and countable $K \subseteq L$,

$$a \vee (\wedge K) = \wedge\{\, a \vee k \;:\; k \in K \,\}.$$

NOTATION 2.2. For a δ-frame L, we denote as follows:

$$\begin{aligned} Fcov(L) =& \{A \in cov(L) : \text{there is a finite cover } B \\ & \qquad \text{with } B \leq A\}, \\ Ccov(L) =& \{A \in cov(L) : \text{there is a countable cover } B \\ & \qquad \text{with } B \leq A\}, \\ Count(L) =& \{A \in L : A \text{ is a countable subset of } L\}. \end{aligned}$$

REMARK 2.3

(1) In a complete lattice L, $a \vee (\wedge K) \leq \wedge\{a \vee k : k \in K\}$ holds for any $K \subseteq L$ and $a \in L$, because $a \leq a \vee k$ for all $k \in K$ implies $\wedge K \leq \wedge\{a \vee k : k \in K\}$, hence $a \vee (\wedge K) \leq \wedge\{a \vee k : k \in K\}$.

(2) Every complete chain L is a δ-frame, because for any $a \in L$ and $K \in Count(L)$,

(i) if $a \leq k$ for all $k \in K$, then $a \leq \wedge K$, hence

$$a \vee (\wedge K) = \wedge K = \wedge\{a \vee k : k \in K\}$$

(ii) if there is $k_0 \in K$ with $k_0 \leq a$, then $\wedge K \leq k_0 \leq a$, hence

$$a \vee (\wedge K) = a = a \vee k_0 \geq \wedge\{a \vee k : k \in K\}$$

Thus by i), ii) and (1), L is a δ-frame.

(3) Every complete Boolean algebra is a δ-frame; hence the frame of regular open subsets of R is a δ-frame. To show this, let L be a complete Boolean algebra. Then for $a \in L$ and $K = \{x_i : i \in I, I \text{ is a countable set}\} \in Count(L)$,

$$\begin{aligned} x_i &= 0 \vee x_i \\ &= (a \wedge a') \vee x_i \\ &= (a \vee x_i) \wedge (a' \vee x_i), \end{aligned}$$

hence

$$\begin{aligned} \wedge K &= \wedge\{(a \vee x_i) \wedge (a' \vee x_i) : x_i \in K\} \\ &= (\wedge\{a \vee x_i : x_i \in K\}) \wedge (\wedge\{a' \vee x_i : x_i \in K\}), \end{aligned}$$

and hence

$$\begin{aligned} a \vee (\wedge K) =& [a \vee (\wedge\{a \vee x_i : x_i \in K\})] \\ & \wedge [a \vee (\wedge\{a' \vee x_i : x_i \in K\})] \\ =& [a \vee (\wedge\{a \vee x_i : x_i \in K\})] \wedge e \\ \geq& \wedge \{a \vee x_i : x_i \in K\}. \end{aligned}$$

REMARK 2.4. Every δ-frame is a frame, but a frame need not be a δ-frame. In fact, the open set lattice $C_f(N)$, where $C_f(N)$ is the cofinite topology on the set of natural numbers N, is not a δ-frame but a frame. Because for $K = \{N - \{m\} : m \text{ is a positive odd integer }\}$ and $a = N - \{2\}$,

$$a = a \vee (\wedge K) \neq \wedge\{a \vee k : k \in K\} = e,$$

where $\wedge K = int(\cap K)$ and $\vee K = \cup K$. But the open set lattice $D(N)$ is a δ-frame, where $D(N)$ is the discrete topology on N. If X is finite, then the open set lattice $\Omega(X)$ with any topology on X is a δ-frame.

REMARK 2.5. Let L be a δ-frame. For $A_n \subseteq L$ and any $n \in N$, let $B = \{a_1 \wedge \cdots \wedge a_n \wedge \cdots : a_n \in A_n$ for each $n \in N\}$. Then we have:

(1) $\vee B = (\vee A_1) \wedge (\vee A_2) \wedge \cdots \wedge (\vee A_n) \wedge \cdots$.

(2) For any $A_n \in cov(L)$, B is the greatest lower bound of A_n for each $n \in N$ in $(cov(L), \leq)$. That is,

$$\wedge_{n \in N} A_n = \{a_1 \wedge \cdots \wedge a_n \wedge \cdots : a_n \in A_n \text{ for each } n \in N\}.$$

Proof. (1) Since L is a δ-frame, L is distributive under a countable meets and arbitrary joins, hence

$$\begin{aligned} &\vee \{a_1 \wedge \cdots \wedge a_n \wedge \cdots : a_n \in A_n \text{ for each } n \in N\} \\ &= \wedge \{\vee A_1, \cdots, \vee A_n, \cdots\} \\ &= (\vee A_1) \wedge \cdots \wedge (\vee A_n) \wedge \cdots . \end{aligned}$$

(2) Note that B is a cover of L, because

$$\begin{aligned} \vee(B) &= \vee\{a_1 \wedge \cdots \wedge a_n \wedge \cdots : a_n \in A_n, n \in N\} \\ &= (\vee A_1) \wedge \cdots \wedge (\vee A_n) \wedge \cdots \\ &= e. \end{aligned}$$

Clearly $B \leq A_n$ for any $n \in N$. Suppose there is C with $C \leq A_n$ for any $n \in N$. Then there is $c \in C$ with $c \leq a_n$ for some $a_n \in A_n$ for each $n \in N$. Hence

$$c \leq a_1 \wedge \cdots \wedge a_n \wedge \cdots \quad \text{for some } a_n \in A_n, \ n \in N.$$

Thus $C \leq B$. □

PROPOSITION 2.6. *Let L be a δ-frame. Then $cov(L), Fcov(L)$, and $Ccov(L)$ are δ-filters.*

Proof. By the above Remark 2.5, it is trivial. □

3. Almost Lindelöf δ-frames and Lindelöf δ-frames

Whenever C is a cover of L, $C - \{0\}$ is also a cover of L. So we will assume that C does not contain 0.

DEFINITION 3.1. Let L be a frame, then we say:

(1) L is said to be *almost Lindelöf* if for any $C \in cov(L)$, there is $K \in Count(C)$ with $(\vee K)^* = 0$.

(2) L is said to be *Lindelöf* if for any $C \in cov(L)$, there is $K \in Count(C)$ with $\vee K = e$.

REMARK 3.2. In a δ-frame L and for any $K \in Count(L)$, we have:

(1) Every compact frame is Lindelöf. But the open set lattice $C_c(N)$ of the cofinite topology on N is a Lindelöf frame but not a compact frame.

(2) $(\vee K)^* = \wedge\{a^* : a \in K\}$. Because

$$\begin{aligned}(\vee K) \wedge (\wedge\{a^* : a \in K\}) \\ &= \vee\{a \wedge (\wedge\{a^* : a \in K\}) : a \in K\} \\ &\leq \vee\{a \wedge a^* : a \in K\} \\ &= 0\end{aligned}$$

and for $y \in L$ with $(\vee K) \wedge y = 0$,

$$(\vee K) \wedge y = \vee\{a \wedge y : a \in K\} = 0.$$

Then $a \wedge y = 0$ for all $a \in K$, and then $y \leq a^*$ for all $a \in K$, hence $y \leq \wedge\{a^* : a \in K\}$.

(3) $(\wedge K)^* \neq \vee\{a^* : a \in K\}$ in general, see Remark 1.7.(5).

In a frame L, L is *almost compact* if and only if for any filter F in L, $\vee\{x^* : x \in F\} \neq e$ ([4],[8]). In a δ-frame L, we get a similar result as following :

THEOREM 3.3. *Let L be a δ-frame, then the followings are equivalent:*

(1) *L is almost Lindelöf.*
(2) *For any δ-filter F in L, $\vee\{x^* : x \in F\} \neq e$.*

Proof. (1) $\Rightarrow$ (2) Suppose on the contrary that there is a δ-filter F in L with $\vee\{x^* : x \in F\} = e$. So there is $K \in Count(F)$ with $(\vee\{y^* : y \in K\})^* = 0$. Note that $(\vee K)^* = \wedge\{x^* : x \in K\}$ by the above Remark 3.2.(2). So

$$(\vee\{y^* : y \in K\})^* = \wedge\{y^{**} : y \in K\} = 0.$$

Note that $x \wedge x^* = 0$ implies $x \leq x^{**}$. Then

$$\wedge\{y : y \in K\} \leq \wedge\{y^{**} : y \in K\} = 0 \text{ implies } \wedge\{y : y \in K\} = 0,$$

which contradicts to the fact that δ-filter F does not contain 0.
(2) $\Rightarrow$ (1) Suppose on the contrary that there is $C \subseteq L$ such that $\vee C = e$, but for any countable $K \subseteq C$,

$$(\vee K)^* = \wedge\{x^* : x \in K\} \neq 0.$$

Let $U = \{y \in L :$ there is a countable K in C with $\wedge\{x^* : x \in K\} \leq y\}$. Then we have the followings:

(i) If $0 \in U$, then there is a $K \in Count(C)$ with $\wedge\{x^* : x \in K\} \leq 0$. Hence $\wedge\{x^* : x \in K\} = 0$, which is a contradiction.
(ii) Let $y \in U$ and $z \geq y$, then there is a $K \in Count(C)$ with $\wedge\{x^* : x \in K\} \leq y \leq z$ and then there is a $K \in Count(C)$ with $\wedge\{x^* : x \in K\} \leq z$. Thus $z \in U$; hence $U = \uparrow U$.
(iii) Let $P = \{y_n : n \in N\} \in Count(U)$, then there is a $K_n \in Count(C)$ with $\wedge\{x^* : x \in K_n\} \leq y_n$ for each $n \in N$.

Let $K = \cup\{K_n : n \in N\}$, then $K \in Count(C)$ and

$$\wedge\{x^* : x \in K\} \leq \wedge\{x^* : x \in K_n\} \leq y_n$$

for each $n \in N$. Thus

$$\wedge\{x^* : x \in K_n\} \leq \wedge\{y_n : n \in N\} = \wedge P,$$

hence $\wedge P \in U$. By (i), (ii) and (iii) U is a δ-filter in L, hence

$$\vee\{y^* : y \in U\} \neq e.$$

Since $\{x^* : x \in C\} \subseteq U$, we have

$$e = \vee C \leq \vee\{x^{**} : x \in C\} \leq \vee\{y^* : y \in U\}.$$

Thus $\vee\{y^* : y \in U\} = e$, which is again a contradiction. □

THEOREM 3.4. *Every regular Lindelöf δ-frame is normal.*

Proof. Let L be a regular Lindelöf δ-frame and $a \vee b = e$. Then

$$a = \vee\{x \in L : x \prec a\},$$
$$b = \vee\{y \in L : y \prec b\}$$

and then

$$e = a \vee b = (\vee\{x \in L : x \prec a\}) \vee (\vee\{y \in L : y \prec b\}).$$

Since L is Lindelöf δ- frame, there are countable cover $\{x_i : i \in I\}$ and countable cover $\{y_k : k \in I\}$ such that $x_i \prec a$ for any $i \in I$, $y_k \prec b$ for any $k \in I$, and $(\vee\{x_i : i \in I\}) \vee (\{y_k : k \in I\}) = e$. Then $x_i^* \vee a = e$ for any $i \in I$ and $y_k^* \vee b = e$ for any $k \in I$. Hence

$$(\wedge\{x_i^* : i \in I\}) \vee a = \wedge\{x_i^* \vee a : i \in I\} = e,$$
$$(\wedge\{y_k^* : k \in I\}) \vee b = \wedge\{y_k^* \vee b : k \in I\} = e.$$

Moreover,

$$\begin{aligned}(\wedge\{x_i^* : i \in I\})\wedge(\wedge\{y_k^* : k \in I\} \\ &= (\vee\{x_i : i \in I\})^* \wedge (\vee\{y_k : k \in I\})^* \\ &= [(\vee\{x_i : i \in I\}) \vee (\{y_k : k \in I\})]^* \\ &= e^* = 0.\end{aligned}$$

Thus L is normal. □

PROPOSITION 3.7. *A δ-filter F in a frame L is clustered if and only if $\vee\{x^* : x \in F\} \neq e$.*

Proof. By Proposition 1.12, it is trivial. □

COROLLARY 3.8. *For a δ-frame L, the followings are equivalent:*

(1) *L is almost Lindelöf.*
(2) *Every δ-filter in L is clustered.*

DEFINITION 3.9. Let L and M be δ-frames and $f : L \to M$ a map.

(1) f is called a *δ-homomorphism* if f preserve arbitrary joins and countable meets.
(2) A δ-homomorphism f is called *dense* if $f(x) = 0$ implies $x = 0$.
(3) A δ-homomorphism f is called *codense* if $f(x) = e$ implies $x = e$.
(4) A δ-homomorphism f is called a *δ-isomorphism* if f is a bijection.

PROPOSITION 3.10. *Let L and M be δ-frames and $f : L \to M$ a δ-homomorphism, then we have:*

(1) *$f(0) = 0$.*
(2) *$f(e) = e$.*
(3) *If f is dense, then $f(a)^* \geq f(a^*)$ for all $a \in L$.*

In particular, $f(a)^* = f(a^*)$ *if* f *is onto dense.*

Proof. (1) $f(0) = f(\vee\emptyset) = \vee f(\emptyset) = \vee\emptyset = 0$.
(2) $f(e) = f(\wedge\emptyset) = \wedge f(\emptyset) = \wedge\emptyset = e$.
(3) If f is dense, then

$$\begin{aligned} f(a)^* &= \vee\{m \in M : m \wedge f(a) = 0\} \\ &\geq \vee\{f(b) \in M : f(b) \wedge f(a) = 0\} \cdots (*) \\ &= \vee\{f(b) \in M : f(b \wedge a) = 0\} \\ &= \vee\{f(b) \in M : b \wedge a = 0\}, \text{ since } f \text{ is dense} \\ &= f(\vee\{b \in L : b \wedge a = 0\} \\ &= f(a^*). \end{aligned}$$

If f is onto, then the inequality $(*)$ becomes equality. □

THEOREM 3.11. *Let* L *and* M *be* δ*-frames and* $f : L \to M$ *is a dense* δ*-homomorphism.*

(1) *If* M *is almost Lindelöf, then so is* L*.*
(2) *If* f *is onto and codense, then* M *is almost Lindelöf if* L *is almost Lindelöf.*

Proof. (1) Take any $S \in cov(L)$, then $\vee S = e$, hence

$$f(e) = e = f(\vee S) = \vee f(S) = \vee\{f(s) : s \in S\}$$

and since M is almost Lindelöf, $f(S) \in cov(M)$ implies there is a $K \in Count(S)$ with $(\vee f(K))^* = 0$. Then

$$\begin{aligned} (\vee\{f(s) : s \in K\})^* &= \wedge\{f(s)^* : s \in K\} \\ &\geq \wedge\{f(s^*) : s \in K\} \\ &= f(\wedge\{s^* : s \in K\}) \\ &= 0, \end{aligned}$$

hence $\wedge\{s^* : s \in K\} = 0$, since f is dense. Thus $(\vee K)^* = \wedge\{s^* : s \in K\} = 0$, hence L is almost Lindelöf.
(2) Take any $T \in cov(M)$, then there is an $S \subseteq L$ with $f(S) = T$ since f is onto, hence $e = \vee T = \vee f(S) = f(\vee S)$ implies $\vee S = e$, since f is codense. So, there is $K \in Count(S)$ with $(\vee K)^* = 0$. Consider $(\vee K)^* = \wedge\{k^* : k \in K\}$ and $f(K) \in Count(T)$, let $W = f(K)$, then

$$\begin{aligned}(\vee W)^* = (\vee f(K))^* &= \wedge\{f(k)^* : k \in K\} \\ &= f(\wedge\{k^* : k \in K\}) \\ &= f(0) = 0.\end{aligned}$$

□

COROLLARY 3.12. *Let L and M be δ-frames and let $f : L \to M$ be a δ-isomorphism which is dense and codense. Then L is Lindelöf if and only if M is Lindelöf.*

REFERENCES

1. R. Balbes and Ph. Dwinger, *Distributive Lattices*, University of Missouri Press, Columbia, 1974.
2. G. Birkhoff, *Lattice Theory*, Amer. Math. Soc. Colloq. Publ. 3rd ed., 1967 Providence.
3. A. Heyting, *Die formalen Regeln der intuitionistischen Logik*, Sitzungsberichte der Preussichen Akademie der Wissenschaften, phys. Mathem. Klasse (1930), 42-56.
4. S. S. Hong, *Simple extensions of frames*, Proc. Recent Devel. of Gen. Top. and its Appl., Math. Research, Akademia Verlag, Berlin **67** (1992), 156-159.
5. S. S. Hong, *Convergence in Frames*, Kyungpook Math. J. **35** (1995), 85-91.
6. B. S. In, *A study on σ-ideals and σ-frames*, Doctorate thesis, Korea University (1987), 22-24.
7. P. T. Johnstone, *Stone Space*, Cambridge University Press, Cambridge, 1982.
8. J. Paseka and B. Šmarda, *T_2-frames and almost compact frames*, Czech. Math. J. **42** (1992), 385-402.

DEPARTMENT OF MATHEMATICS
CHUNGBUK NATIONAL UNIVERSITY
CHEONGJU 361-763, KOREA