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THE PERRON AND VARIATIONAL INTEGRALS

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ABSTRACT. In this paper, we give a direct proof that the Perron and variational integrals are equivalent.

1. Introduction

In 1914, O. Perron developed an extension of the Lebesgue integral and showed that his integral also had the property that every derivative was integrable.

There is another integral which is an extension of the Lebesgue integral. It is known as the variational integral. The variational integral represents a transition between the Henstock and Perron integrals.

In this paper, we show that the Perron and variational integrals are equivalent.

The first step is to introduce the notion of *upper derivate* $\overline{D}F(x)$ and *lower derivate* $\underline{D}F(x)$ of a function $F : [a, b] \to R$ at $x \in [a, b]$ defined by

$$\overline{D}F(x) = \lim_{\delta \to 0^+} \sup \left\{ \frac{F(y) - F(x)}{y - x} \;\middle|\; 0 < |y - x| < \delta \right\},$$

$$\underline{D}F(x) = \lim_{\delta \to 0^+} \inf \left\{ \frac{F(y) - F(x)}{y - x} \;\middle|\; 0 < |y - x| < \delta \right\}.$$

Let $f : [a, b] \to R_e$ be the extended real-valued function on $[a, b]$. A function $U : [a, b] \to R$ is a *major function* of f on $[a, b]$ if $\underline{D}U(x) > -\infty$ and $\underline{D}U(x) \geq f(x)$ for all $x \in [a, b]$. A function $V : [a, b] \to R$ is a *minor function* of f on $[a, b]$ if $\overline{D}V(x) < \infty$ and $\overline{D}V(x) \leq f(x)$ for all $x \in [a, b]$.

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2. The euivalence of the Perron and variational Integrals

In this section, we define the Perron integral and the variational integral, and we show that the Perron integral is equivalent to the variational integral.

DEFINITION 2.1. A function $f : [a,b] \to R$ is *Perron integrable* on $[a,b]$ if f has at least one major and one minor function on $[a,b]$ and the numbers

$$\inf\{U_a^b : U \text{ is a major function of } f \text{ on } [a,b]\},$$

$$\sup\{V_a^b : V \text{ is a major function of } f \text{ on } [a,b]\}$$

are equal, where $U_a^b = U(b) - U(a)$ and $V_a^b = V(b) - V(a)$. This common value is the *Perron integral* of f on $[a,b]$ and will be denoted by $\int_a^b f$. The f is Perron integrable on a measurable set $E \subseteq [a,b]$ if $f\chi_E$ is Perron integrable on $[a,b]$.

The following theorem is an immediate consequence of the definition and its proof is omitted.

THEOREM 2.2. *A function $f : [a,b] \to R_e$ is Perron integrable on $[a,b]$ if and only if for each $\epsilon > 0$ there exist a major function U and a minor function V of f on $[a,b]$ such that $U_a^b - V_a^b < \epsilon$.*

Let $\delta(\cdot)$ be a positive function defined on the interval $[a,b]$. A *tagged interval* $(x, [c,d])$ consists of an interval $[c,d] \subseteq [a,b]$ and a point $x \in [c,d]$. The tagged interval $(x,[c,d])$ is *subordinate to* δ if $[c,d] \subseteq (x - \delta(x), x + \delta(x))$.

DEFINITION 2.3. A function $f : [a,b] \to R$ is *variational integrable* on $[a,b]$ if there exists a function $F : [a,b] \to R$ with the following property: for each $\epsilon > 0$ there exist a nondecreasing function ϕ defined on $[a,b]$ and a positive function δ defined on $[a,b]$ such that

$$\phi(b) - \phi(a) < \epsilon \text{ and } |f(x)(d-c) - (F(d) - F(c))| \leq \phi(d) - \phi(c)$$

whenever $(x, [c, d])$ is a tagged interval in $[a, b]$ that is subordinate to δ.

In [4], it was proved that the Denjoy, Perron, Henstock, and variational integrals are equivalent.

Now we will give a direct proof that the Perron and variational integrals are equivalent.

THEOREM 2.4. *A function $f : [a, b] \to R$ is Perron integrable on $[a, b]$ if and only if it is variational integrable on $[a, b]$.*

Proof. Let f be variational integrable on $[a, b]$ and let F be its variational integral. Given $\epsilon > 0$, choose ϕ and δ according to the definition of the variational integral. We show that $U = F + \phi$ is a major function of f and that $V = F - \phi$ is a minor function of f on $[a, b]$. If $0 < x - y < \delta(x)$, then a tagged interval $(x, [y, x])$ is subordinate to δ. From the variational integrability of f, we have

$$|f(x)(x - y) - (F(x) - F(y))| \leq \phi(x) - \phi(y)$$

and

$$f(x) \leq \frac{F(x) - F(y)}{x - y} + \frac{\phi(x) - \phi(y)}{x - y} = \frac{U(x) - U(y)}{x - y}.$$

Since this inequality is valid for $0 < y - x < \delta(x)$, we have

$$-\infty < f(x) \leq \underline{D}U(x).$$

This implies that U is a major function of f. The proof that V is a minor function of f on $[a, b]$ is quite similar. In addition,

$$U_a^b - V_a^b = 2\{\phi(b) - \phi(a)\} < 2\epsilon.$$

Hence, f is Perron integrable on $[a, b]$.

Now suppose that f is Perron integrable on $[a, b]$. Let $\epsilon > 0$ and let $\eta = min\{\frac{\epsilon}{2(b-a)}, \frac{\epsilon}{2}\}$. By definition, there exist a major function U and a minor function V of f on $[a, b]$ such that $U_a^b - V_a^b < \eta$. Then $\phi(x) = U(x) - V(x) + \eta x$ is a nondecreasing function and $\phi(b) - \phi(a) = U_a^b - V_a^b + \eta(b-a) < \epsilon$.

Since $\overline{D}V \le f \le \underline{D}U$ on $[a, b]$, for each $x \in [a, b]$ there exists $\delta(x) > 0$ such that

$$\frac{U(y) - U(x)}{y - x} \ge f(x) - \eta \quad \text{and} \quad \frac{V(y) - V(x)}{y - x} \le f(x) + \eta$$

whenever $0 < |y - x| < \delta(x)$ and $y \in [a, b]$. Let $F(x) = (P) \int_a^x f$. If $(x, [c, d])$ is a tagged interval in $[a, b]$ that is subordinate to δ, then

$$\begin{aligned}
f(x)(d-c) &- (F(d) - F(c)) \\
&= \{f(x)(x-c) - (F(x) - F(c))\} \\
&\quad + \{f(x)(d-x) - (F(d) - F(x))\} \\
&\le \{U_c^x + \eta(x-c) - (F(x) - F(c))\} \\
&\quad + \{U_x^d + \eta(d-x) - (F(d) - F(x))\} \\
&\le (U_c^x - V_c^x) + (U_x^d - V_x^d) + \eta(d-c) \\
&= \{U(d) - V(d) + \eta d\} - \{U(c) - V(c) + \eta c\} \\
&= \phi(d) - \phi(c).
\end{aligned}$$

Similarly, we have

$$f(x)(d-c) - (F(d) - F(c)) \ge \phi(c) - \phi(d).$$

From these two inequalities, we have

$$|f(x)(d-c) - (F(d) - F(c))| \le \phi(d) - \phi(c).$$

Hence f is variational integrable on $[a, b]$. □

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