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THE EQUIVALENCE OF PERRON, HENSTOCK AND VARIATIONAL STIELTJES INTEGRALS

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ABSTRACT. In this paper, we verify that Perron, Henstock and variational Stieltjes integrals are all equivalent. That is, a function which is integrable in one sense is integrable in the other sense and their values are all equal.

1. Introduction

In 1914, O. Perron developed an extension of the Lebesgue integral and showed that this integral also had the property that every derivative was integrable. This integral was designed to overcome the deficiencies of the Lebesgue integral.Another integral developed for the same purpose is the Henstock integral. The definition of the Henstock integral is very similar to the definition of the Riemann integral as we will see. There is other integral that is defined without measure. It is known as the variational integral and definition varies from author to author depending on the situation. The variational integral represents a transition between the Henstock and Perron integrals. It avoids Riemann sums and derivates. On this situation, it is very natural that we consider the Stieltjes type integral for the integrals. These are the Perron, Henstock, and variational Stieltjes integrals. In this paper, we verify that these integrals are all equivalent. That is, a function which is integrable in one sense is integrable in the other

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sense. Now, we give basic definitions and results for us to prove the main results.

DEFINITION 1.1. Let f and G be finite valued functions on $[a, b]$ and G be strictly increasing on $[a, b]$. Then,

(a) A function $U : [a, b] \to R$ is a *major function* of f with respect to G on $[a, b]$ if

$$\underline{D}U^G(x) > -\infty \text{ and } \underline{D}U^G(x) \geq f(x) \text{ for all } x \in [a, b],$$

$$\text{where } \underline{D}U^G(x) = \lim_{\delta \to 0+} \inf\left\{\frac{U(y)-U(x)}{G(y)-G(x)} : 0 < |x-y| < \delta\right\}.$$

(b) A function $V : [a, b] \to R$ is a *minor function* of f with respect to G on $[a, b]$ if

$$\bar{D}V^G(x) < +\infty \text{ and } \bar{D}V^G(x) \leq f(x) \text{ for all } x \in [a, b],$$

$$\text{where } \bar{D}V^G(x) = \lim_{\delta \to 0+} \sup\left\{\frac{V(y)-V(x)}{G(y)-G(x)} : 0 < |x-y| < \delta\right\}.$$

DEFINITION 1.2. Let f and G be finite valued functions on $[a, b]$ and G be strictly increasing on $[a, b]$. Then, f is *Perron-Stieltjes integrable* with respect to G on $[a, b]$ if f has at least one major function and one minor function with respect to G on $[a, b]$ and the numbers

$$\inf\left\{U_a^b : U \text{ is a major function of } f \text{ with respect to } G \text{ on } [a, b]\right\},$$
$$\sup\left\{V_a^b : V \text{ is a minor function of } f \text{ with respect to } G \text{ on } [a, b]\right\}$$

are equal, where $U_a^b = U(b) - U(a)$ and $V_a^b = V(b) - V(a)$. This common value is called *the Perron-Stieltjes(P-S) integral* of f with repect to G on $[a, b]$. We denote the common value as $(P)\int_a^b f dG$.

THEOREM 1.3. (Cauchy Criterion for the Perron-Stieltjes Integral) *A function* $f : [a, b] \to R$ *is P-S integrable with respect to* G *on* $[a, b]$ *if and only if for each* $\epsilon > 0$ *there exists a major function* U *and a minor function* V *of* f *with respect to* G *on* $[a, b]$ *such that* $U_a^b - V_a^b < \epsilon$.

In order to define the Henstock-Stielties(H-S) integral, it is necessary to specify the terms that are allowed in the Riemann-Stieltjes sums. We begin by looking at certain types of tagged partitions. Pay close attention to the notation and terminology.

DEFINITION 1.4. Let $\delta(x)$ be a positive function defined on $[a, b]$. *A tagged interval* $(x, [c, d])$ consists of an interval $[c, d] \subset [a, b]$ and a point $x \in [c, d]$. The tagged interval $(x, [c, d])$ is *subordinate* to δ if $[c, d] \subset (x - \delta(x), x + \delta(x))$. The letter P will be used to denote finite collections of nonoverlapping tagged intervals. Let $P = \{(x_i, [c_i, d_i]) : 1 \le i \le n\}$ be such a collection in $[a, b]$. We adopt the following terminology.

(a) The points x_i are the tags of P and the intervals $[c_i, d_i]$ are the intervals of P.
(b) If $(x_i, [c_i, d_i])$ is subordinate to δ for each i, then P is subordinate to δ.
(c) If P is subordinate to δ and $[a, b] = \cup_{i=1}^{n} [c_i, d_i]$, then P is a tagged partition of $[a, b]$ that is subordinate to δ.

Let $P = \{(x_i, [c_i, d_i]) : 1 \le i \le n\}$ be a finite collection of nonoverlapping tagged intervals in $[a, b]$, let $f : [a, b] \to R$ be a finite valued function defined on $[a, b]$ and let F be a function defined on the subintervals of $[a, b]$. We will use the following notations,

$$f^G(P) = \sum_{i=1}^{n} f(x_i)(G(d_i) - G(c_i)),$$

$$F(P) = \sum_{i=1}^{n} F([c_i, d_i]) = \sum_{i=1}^{n} (F(d_i) - F(c_i))$$

and $F(x) = \int_a^x f dG$ will always be treated as a function of intervals when defined on tagged partitions of $[a, b]$, that is,

$$F([c, d]) = F(d) - F(c) = \int_a^d f dG - \int_a^c f dG.$$

DEFINITION 1.5. Let f and G be two finite valued fuctions on $[a, b]$. Then f is *Henstock-Stieltjes(H-S) integrable* with respect to G on $[a, b]$ if there exists a real number L with the following property: for every $\epsilon > 0$, there exists a positive function δ on $[a, b]$ such that $|f^G(P) - L| < \epsilon$ whenever P is a tagged partition of $[a, b]$ that is subordinate to δ. We denote $L = (H) \int_a^b f dG$.

DEFINITION 1.6. Let f and G be finite valued functions on $[a, b]$. Then f is *variational Sieltjes(V-S) integrable* with respect to G on $[a, b]$ if there exists a function $F : [a, b] \to R$ with the following property: for each $\epsilon > 0$, there exists nondecreasing function ϕ defined on $[a, b]$ and a positive function δ defined on $[a, b]$ such that $\phi(b) - \phi(a) < \epsilon$ and

$$|f(x)(G(d) - G(c)) - (F(d) - F(c))| \leq \phi(d) - \phi(c)$$

whenever $(x, [c, d])$ is a tagged interval in $[a, b]$ that is subordinate to δ. We denote the integral as $(V) \int_a^b f dG = F(b) - F(a)$.

The following lemma is a simple modification of Saks-Henstock lemma.

LEMMA 1.7. (Henstock-Stieltjes Lemma) *Let f and G be two functions defined on $[a, b]$ and let $\epsilon > 0$. Assume that f is H-S integrable with respect to G on $[a, b]$ and let $F(x) = \int_a^x f dG$ for each $x \in [a, b]$. Suppose that δ is a positive function on $[a, b]$ such that*

$|f^G(P) - F(P)| < \epsilon$ whenever P is a tagged partition of $[a,b]$ that is subordinate to δ. If $P_0 = \{(x_i, [c_i, d_i]) : 1 \le i \le n\}$ is subordinate to δ, then

$$|f^G(P_0) - F(P_0)| \le \epsilon,$$
$$\sum_{i=1}^{n} |f(x_i)(G(d_i) - G(c_i)) - (F(d_i) - F(c_i))| \le 2\epsilon.$$

The following definition is related to the variational integral.

DEFINITION 1.8. Let Φ be a function defined on the intervals on $[a,b]$. The function Φ is *superadditive* if

$$\Phi([u,v]) + \Phi([v,w]) \le \Phi([u,w])$$

whenever $a \le u < v < w \le b$.

2. Main Results

THEOREM 2.1. *Let f be a finite valued function on $[a,b]$ and G be finite and strictly increasing on $[a,b]$. Then, if f is P-S integrable on $[a,b]$, then f is H-S integrable on $[a,b]$ and the integrals are equal.*

Proof. The proof of this theorem is very similar to the proof of [1, Theorem 11.5]. The only difference is that G is strictly increasing on $[a,b]$. This condition guarantees that $G(y) - G(x)$ is positive for all $y > x$. So in the process of the proof, the directions of inequalities are not changed. □

THEOREM 2.2. *Let f be a finite valued function on $[a,b]$ and G be finite and strictly increasing on $[a,b]$. If f is H-S integrable on $[a,b]$, then f is P-S integrable on $[a,b]$ and the integrals are equal.*

Proof. Let $\epsilon > 0$. By definition, there exists a positive function δ on $[a,b]$ such that $|f^G(P) - (H)\int_a^b f dG| < \epsilon$ whenever P is a tagged partition of $[a,b]$ that is subordinate to δ. For each $x \in [a,b]$, let

$$\begin{aligned} U(x) = \sup\{f^G(P) : P &\text{ is a tagged partition of } [a,x] \\ &\text{that is subordinate to } \delta\}, \\ V(x) = \inf\{f^G(P) : P &\text{ is a tagged partition of } [a,x] \\ &\text{that is subordinate to } \delta\} \end{aligned}$$

and let $U(a) = 0 = V(a)$. By the Henstock-Stieltjes Lemma, the functions U and V are finite valued on $[a,b]$. Then we can verify that U and V are major and minor function of f with respect to G on $[a,b]$ as follows:
Fix a point $c \in [a,b]$. For $x \in (c, c+\delta(c)) \cap [a,b]$ and for any tagged partition P of $[a,c]$ that is subordinate to δ, we fined that

$$U(x) \geq f^G(P) + f(c)(G(x) - G(c))$$

and it follows that

$$U(x) \geq U(c) + f(c)(G(x) - G(c)).$$

For $x \in (c - \delta(c), c) \cap [a,b]$ and for any tagged partition P of $[a,x]$ that is subordinate to δ,

$$U(c) \geq f^G(P) + f(c)(G(c) - G(x))$$

and it follows that

$$U(c) \geq U(x) + f(c)(G(c) - G(x)).$$

This shows that

$$\frac{U(x) - U(c)}{G(x) - G(c)} \geq f(c)$$

for all $x \in (c - \delta(c), c + \delta(c)) \cap [a, b]$ with $x \neq c$ and hence it follows that

$$\underline{D}U^G(c) \geq f(c) > -\infty.$$

Thus the function U is a major function of f with respect to G on $[a, b]$. Similarly we can verify that V is a minor function of f with respect to G on $[a, b]$. The rest of the proof is almost identical to that of [1, Theorem 11.6]. □

The following theorem is needed to prove the equivalence of those three Stieltjes integrals.

THEOREM 2.3. *Let f and G be two functions defined on $[a, b]$. Then, f is H-S integrable on $[a, b]$ if and only if there exists a function $F : [a, b] \to R$ with the following property: for each $\epsilon > 0$ there exists a superadditive interval function Φ defined on $[a, b]$ and a positive function δ defined on $[a, b]$ such that $\Phi([a, b]) \leq \epsilon$ and*

$$|f(x)(G(d) - G(c)) - (F(d) - F(c))| \leq \Phi([c, d])$$

whenever $(x, [c, d])$ is a tagged interval in $[a, b]$ that is subordinate to δ.

Proof. The proof of this theorem is similar to that of [1, Theorem 11.9]. The proof is obtained as we simply replace $d_i - c_i$ by $G(d_i) - G(c_i)$ in the proof of [1, Theorem 11.9]. □

We show in the following theorem the equivalence of H-S integral and V-S integral.

THEOREM 2.4. *Let f and G be finite valued functions on $[a, b]$. Then f is H-S integrable with respect to G on $[a, b]$ if and only if it is V-S integrable with respect to G on $[a, b]$.*

Proof. The proof of this theorem is almost identical to that of [1, Theorem 11.10]. □

Thus we obtain the following theorem.

THEOREM 2.5. *If f is a finite valued function on $[a,b]$ and G is finite and strictly increasing on $[a,b]$, then P-S, H-S and V-S integrals are all equivalent.*

REFERENCES

1. R. A. Gordon, *The integrals of Lebesgue, Denjoy, Perron, and Henstock*, Amer. Math. Soc., 1994.
2. I. P. Natanson, *Theory of functions of a real variable*, Frederick Ungar Publishing Co, 1964.
3. W. Pfeffer, *A note on the Generallized Riemann integral*, Proc. Amer. Math. Soc **103** (1988), 1161-1166.
4. S. Saks, *Theory of the integrals*, Hafner, New York, 1937.
5. W. Rudin, *Principles of mathematical analysis, Third edition*, McGraw-Hill, New York.
6. Y. Kubota, *Remarks on the Henstock integral of Stieltjes type*, Bull. Fac. Sci. (Ser. A) **17** (1985), 25-29.

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